The preface, entitled "The Translator to the Reader" besides being a very beautiful piece of printing (Figure 2) with its tapered printing at the close, and the beautiful and intricate geometrical decoration, is so full of quaint praise for the subject of mathematics in general and of geometry in particular that I cannot refrain from quoting it in its entirety:

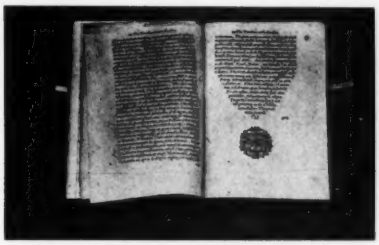

FIGURE 2

FIGURE 3

"THE TRANSLATOR TO THE READER."

"There is (gentle reader) nothing (the Word of God onely set apart) which so much beautifieth and adorneth the soule and minde of mā, as doth the knowledge of good artes and sciences: as the knowledge of naturall and morall Philosophie. The one setteth before our eyes, the creatures of God, both in the heauens aboue, and in the earth beneath: in which as in a glasse, we behold the exceding maiestie and wisedom of God, in adorning and beautifying them as we see: in geuing vnto them such wonderfull and manifolde proprieties, and natural workinges, and that so diuersely and in such varietie: farther in maintaining and conseruing them continually, whereby to praise and adore him, as by S. Paule we are taught. The other teacheth vs rules and preceptes of vertue, how, in common life amongest men, we ought to walke uprightly: what dueties pertaine to our selues, what pertaine to the gouernment or good order both of an housholde, and also of a citie or common wealth. The reading likewise of histories, conduceth not a litle, to the adorning of the soule and minde of man, a study of all men cōmended: by it are seene and knowen the artes and doinges of infinite wise men gone before vs. In histories are contained infinite examples of heroicall vertues to be of vs followed, and horrible examples of vices to be of vs eschewed. Many other artes also there are which beautifie the minde of man: but of all other none do more garnishe and beautifie it, than those artes which are called Mathematicall. Vnto the knowledge of which no man can attaine, without the perfect knowledge and instruction in the principles, groundes, and Elementes of Geometrie. But perfectly to be instructed in them, requireth diligent studie and reading of olde auncient authors. Amongst which, none for a beginner is to be preferred before the most auncient Philosopher Euclide

of Megara. For of all others, he hath in a true methode and iuste order, gathered tigether whatsoeuer any before him had of these Elementes written: inuenting also and adding many thinges of his owne: whereby he hath in due form accomplished the arte: first geuing definitions, principles, and groundes, whereof he deduceth his Propositions or conclusions, in such wonderfull wise, that that which goeth before, is of necessitie required to the proufe of that which followeth. So that without the diligent studie of Euclides Elementes, it is impossible to attaine vnto the perfect knowledge of Geometrie, and consequently of any of the other Mathematicall sciences. Wherefore considering the want and lacke of such good authors hitherto in our Englishe tongue, lamenting also the negligence, and lacke of zeale to their countrey in those of our nation, to whom God hath geuen both knowledge and also abilitie to translate into our tounge, and to publishe abroade such good authors, and bookes (the chiefe instrumentes of all learninges): seing moreouer that many good wittes both of gentlemen and of others of all degrees, much desirous and studious of these artes, and seeking for them as much as they can, sparing no paines and yet frustrate of their intent, by no meanes attaining to that which they seeke: I haue for their sakes, with some charge and great trauail, faithfully translated into our vulgare toũge, and set abroad in Print, this booke of Euclide. Whereunto I haue added easie and plaine declarations and examples by figures, of the definitions. In which book also ye shall in due place finde manifolde additions, Scholies, Annotations, and Inuentions: which I haue gathered out of many of the most famous and chiefe Mathematiciẽs, both of olde time, and in our age: as by diligent reading it in course, ye shall well perceaue. The fruite and gaine which I require for these my paines and trauaile, shall be nothing els, but onely that thou gentle reader, will gratefully accept the same: and that thou mayest thereby receaue some profite: and moreouer to excite and stirre vp others learned, to do the like, and to take pains in that behalfe. By meanes wherof, our Englishe tounge shall no lesse be enriched with good Authors, then are other straunge tounges: as the Dutch, French, Italian, and Spanish: in which are red all good authors in a maner, found amongest the Grekes or Latines. Which is the chiefest cause, that amongest thẽ do florishe so many cunning and skilfull men, in the inuentions of straunge and wonderfull thinges, as in these our daies we see there do. Which fruite and gaine if I attaine vnto, it shall encourage me hereafter, in such like sort to translate, and set abroad some other good authors, both pertaining to religion (as partly I haue already done) and also pertaining to the Mathematicall Artes. Thus gentle reader farewell."

Matters of very considerable interest are contained in the fruitful preface by John Dee, of London, who seems to have been a rather learned man of his day in Mathematics and Science, but who for some reason or other was not in particularly good repute with some of his contemporaries. His preface is addressed "To the Vnfained Lovers of truthe, and constant Studentes of Noble Sciences, John Dee of London, hartily wisheth grace from heauen, and most prosperous successe in all their honest attemptes and exercise." In this article Mr. Dee sets himself to the task of naming and defining all the various subjects which have a mathematical background and then presents a tabular summary the size of four folio pages in which all these are listed in relation to each other. It is a matter of sincere regret that I have been unable to get a satisfactory photograph of this table. Not the least interesting bit of this extensive preface—it covers forty-six folio pages—is a paragraph of invective against those who do not look kindly upon his work, from which I quote in part as an exceedingly good use of the English language in invective:

"A Digression Apologeticall

And for these, and such like maruelous Actes and Feates, Naturally, Mathematically, and Mechanically, wrought and contriued: ought any honest Student, and Modest Christian Philosopher, be counted & called a CONIURER? Shall the folly of Idiotes, and the Mallice of the Scornfull, so much preuaile, that He, who seeketh no worldly gaine or glory at their handes: But onely, of God, the treasor of heauenly wisedome, & knowledge of pure veritie: Shall he (I say) in the meane space, be robbed and spoiled of his honest name and fame? . . . . Will they provoke him, by worde and Printe, likewise, to Note their Names to the World: with their particular deuises, fables, beastly Imaginations, and vnchristenlike slaunders? Well: Well. O (you such) my unkinde Country men. O vnnaturall Countrey men. O vnthankfull Countrey men. O Brainsicke, Rashe, Spitefull, and Disdainfull Countrey men. Why oppresse you me, thus violently, with your slaundering of me: Contrary to Veritie: and contrary to your owne Consciences?"

After considerable more of this delicious raving, he continues with his summary, and finally closes with these humble words: "If Hast, hath caused my poore pen, anywhere, to stumble: You will, (I am sure) in part of recompence, (for my earnest and sincere good will to pleasure you), consider the rockish huge mountaines, and the perilous vnbeaten wayes, which (both night and day, for the while) it hath toyled and labored through, to bryng you this good Newes, and comfortable profe, of Vertues frute. So, I commit you vnto Gods Mercyfull direction, for the rest: hartely besechyng hym, to prosper your

Studyes, and honest Intentes: to his Glory, & the Commodity of our Countrey. Amen.

Written at my poore House
At Mortlake.
Anno. 1570. February 9."

The first volume contains the first nine books of Euclid, printed with beautiful initials, extensive proofs, and many additional notes by Pelitarius, Orontius, Flussates, Campane, Dee, and others. Figure 3 shows the book open at folio 58, displaying on the right the figure and proof of the Pythagorean Theorem essentially as it is given now in practically all modern texts. A note acknowledging Pythagorus as the inventor says: "This most excellent and notable Theoreme was first inuented of the greate philosopher Pithagoras, who for the exceeding joye conceiued by the inuention thereof, offered in sacrifice an Oxe, as recorde Hierone, Proclus, Lycius, and Vitruutus. And it hath bene commōly called of barbarous writers of later time Dulcarnon."

The second volume contains the tenth to fifteenth books of Euclid, with a sixteenth book added by Flussas. The wood-cuts illustrating the solid figures are most beautiful, but one of the most interesting features of the eleventh book is that many of the figures are made of paper and so pasted in the book that they may be opened up to make actual models of the space figures. Figure 4 shows the page of definitions of pyramids, and illustrates with actual models, pyramids with triangular, quadrilateral, and pentagonal bases.

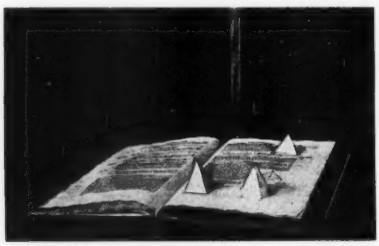

FIGURE 4

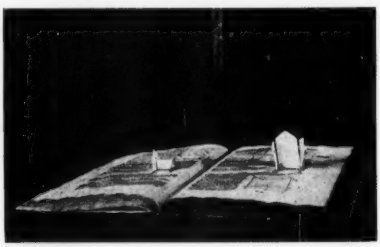

FIGURE 5

Figure 5 shows similar figures to illustrate the proofs of "From a point geuen on high, to draw vnto a ground plaine superficies a perpendicular right line." (Model on Left Page) and "Vnto a playne superficies geuen, and from a poynt in it geuen, to rayse up a perpendicular line." The models to illustrate "If two right lines touching the one the other be parallels to other right lines touching also the one the other and not being in the self same plaine superficies with the two first: the plaine superficies extended by those right lines, are also paral-

lels the one to the other." and "If two parallel playne superficies be cut by some one playne superficies: their common sections are parallel lines," are shown to the left and right, respectively, of Figure 6. Figure 7 is inserted to show the

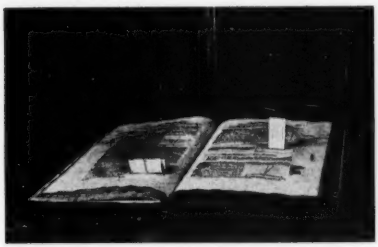

FIGURE 6

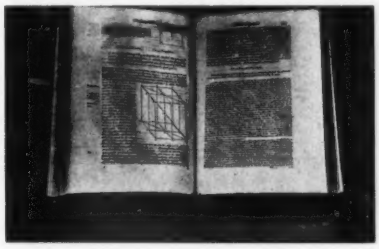

FIGURE 7

beautiful space drawings in the text, and incidentally shows another plane model in the upper left hand corner.

The colophon of this book (Figure 8) shows a very excellent wood-cut of John Day, the printer. Johnson's "Typographia" says (Vol. 1, P. 540) of this colophon: "The portrait of Day, which appears at the head of this article, is perhaps the earliest of our ancient printers, which can be depended upon as

FIGURE 8

genuine. It is taken from the 'Elements of Geometrie of the most auncient Philosopher Euclide of Megara' 1570, Folio; but the original wood cut bears the date of 1562. The print itself has a large cartouche oval frame, with pinks and gilly flowers issuing from the four corners, and immediately round the portrait is a band containing the following words in Roman Capitals:

LIEFE.IS.DEATH. AND DEATH IS.LIEFE:AETATIS.SVAE:XXXX."

In company with this text which brought into our mother tongue the elements of Geometry which today play such a large part in our business and professional life, the library which American University has received from Artemas Martin is rich in other very early mathematical publications. It also contains a collection of the arithmetics published in America from the very earliest dates up to the present century, as well as other historical mathematical material. It will be a great pleasure to make these volumes available to the members of the Mathematical Association to view and use for such research purposes as they might suggest.

THE PHASE RULE

By J. E. TREVOR, Cornell University

For the case of a body constituted of a single component substance and consisting of n homogeneous masses or "phases" coexisting in thermodynamic equilibrium, the famous phase rule of J. Willard Gibbs may be expressed by the statement:

The pressure and temperature of the body, and the specific volumes and specific entropies of its several phases, form a set of $2n+2$ variables of which $3-n$ are independent.

It has long been recognized that this "rule" is merely a conclusion from the circumstance that the $2n+2$ variables are connected by $3n-1$ relations, and that it is not clear whether an arbitrary choice of $3-n$ variables, of which the remaining variables are then functions, can be made. The purpose of the present note is to examine the functional dependence involved.

The necessary data may be assembled as follows. The volumes, entropies, and masses of the several phases are variables,

$$V_i, \quad S_i, \quad M_i, \quad i = 1,2, \cdots, n,$$

which are independent save as M_1 is defined by the equation

$$c = M_1 + M_2 + \cdots + M_n,$$

where the constant c is the mass of the body. The energies of the phases are continuous functions,

$$E_i(V_i,S_i,M_i), \quad i = 1,2, \cdots, n.$$

The volume V, the entropy S, and the energy E of the body are functions defined by the equations,

$$V = \sum V_i, \qquad S = \sum S_i, \qquad E = \sum E_i; \tag{1}$$

and it is required to find necessary and sufficient conditions for the functions V, S, E to satisfy a relation of the form

$$E = F(V,S). \tag{2}$$

By (2), $dE-(\partial F/\partial V)\,dV-(\partial F/\partial S)dS=0$. When this equation is expanded by making use of the definitions (1), whereupon the dependent differential $dM_1=-dM_2-\cdots-dM_n$ is eliminated, necessary conditions are found to be:

$$\begin{aligned}(\partial F/\partial V) &= (\partial E_1/\partial V_1) = (\partial E_2/\partial V_2) = \cdots = (\partial E_n/\partial V_n),\\ (\partial F/\partial S) &= (\partial E_1/\partial S_1) = (\partial E_2/\partial S_2) = \cdots = (\partial E_n/\partial S_n),\\ (\partial E_1/\partial M_1) &= (\partial E_2/\partial M_2) = \cdots = (\partial E_n/\partial M_n).\end{aligned} \tag{3}$$

I am not aware that the sufficiency of these conditions has ever been discussed. The last of the definitions (1) is

$$E = E_1(V_1,S_1,M_1) + E_2(V_2,S_2,M_2) + \cdots + E_n(V_n,S_n,M_n); \tag{4}$$

Here, by the remaining equations (1) and the definition of M_1, let the variables V_1, S_1, M_1 be the functions

$$V_1 = V - V_2 - \cdots - V_n, \quad S_1 = S - S_2 - \cdots - S_n, \quad M_1 = c - M_2 - \cdots - M_n.$$

Then, by differentiating (4) with regard to the subscripted variables, we find

$$\begin{aligned}(\partial E/\partial V_i) &= -(\partial E_1/\partial V_1) + (\partial E_i/\partial V_i), \qquad i = 2, 3, \cdots, n,\\ (\partial E/\partial S_i) &= -(\partial E_1/\partial S_1) + (\partial E_i/\partial S_i), \qquad i = 2, 3, \cdots, n,\\ (\partial E/\partial M_i) &= -(\partial E_1/\partial M_1) + (\partial E_i/\partial M_i), \qquad i = 2, 3, \cdots, n.\end{aligned}$$

By the conditions (3), when the members $\partial F/\partial V$ and $\partial F/\partial S$ are omitted, the second members of these equations are equal to zero. So E contains none of the subscripted variables, and hence is a function of V and S alone. The $3(n-1)$ conditions (3) employed are therefore sufficient.

It is given, further, that each of the n functions E_i is homogeneous of degree one in its variables. The first derivatives of these functions are therefore homogeneous of degree zero, and thus are functions of the respective ratios,

$$v_i = V_i/M_i, \quad s_i = S_i/M_i, \quad i = 1,2, \cdots, n, \tag{5}$$

which are the specific volumes and the specific entropies of the several phases. Now the pressure p and the absolute temperature θ under which the state of the body is a state of thermodynamic equilibrium are continuous functions of V, S, and are defined by the equations $-p=\partial F/\partial V$, $\theta=\partial F/\partial S$. So we have that the conditions (3) subject the $2n+2$ quantities p, θ, v_i, s_i, to $3n-1$

relations. When (3) are satisfied the number of these quantities remaining independent is $(2n+2)-(3n-1)=3-n$.

It is improbable that the conditions (3) can be satisfied when n exceeds 3. When $n=1$ it is desired to know what sets of two of the variables p, θ, v_1, s_1, are functions of the two remaining variables. When $n=2$ it is desired to know what sets of five of the variables $p, \theta, v_1, s_1, v_2, s_2$, are functions of the remaining variable. When $n=3$ it is desired to know whether the quantities p, θ, v_i, s_i are all constants.

The Case $n=1$

We must begin by expressing the conditions (3) in terms of p, θ and the "specific" variables v_i, s_i. Let t be any positive number. Then, the function E_i being homogeneous of degree one, we have

$$E_i(tV_i, tS_i, tM_i) = t \cdot E_i(V_i, S_i, M_i) ;$$

or, putting $t=1/M_i$ and using the definitions (5),

$$E_i = M_i \cdot e_i(v_i, s_i). \tag{6}$$

When expressed by means of this formulation, the conditions (3), for $n=1$, are

$$-p = \partial e_1/\partial v_1, \quad \theta = \partial e_1/\partial s_1, \tag{7}$$

which shall be written

$$f_1 = p + (\partial e_1/\partial v_1) = 0, \quad f_2 = \theta - (\partial e_1/\partial s_1) = 0.$$

When the body is in a one-phase state of thermodynamic equilibrium these equations are satisfied by a set of values of the variables p, θ, v_1, s_1. And, in the neighborhood of this set of values, the functions f_1, f_2 are continuous and possess continuous first derivatives with regard to all their variables. Now, to determine whether, in this neighborhood, a selected two, u, v, of the four variables are single-valued continuous functions of the remaining two variables x, y, we require to determine whether the jacobian $\partial(f_1, f_2)/\partial(u, v)$ is non-vanishing at the point. The jacobians of f_1, f_2 with regard to the elements of each of the six possible pairs of variables are as follows:

$$\partial(f_1,f_2)/\partial(p,\theta) = +1, \qquad \partial(f_1,f_2)/\partial(p,s_1) = -\partial^2 e_1/\partial s_1^2,$$
$$\partial(f_1,f_2)/\partial(v_1,\theta) = +\partial^2 e_1/\partial v_1^2, \qquad \partial(f_1,f_2)/\partial(v_1,s_1) = -\Delta_1,$$
$$\partial(f_1,f_2)/\partial(\theta,s_1) = -\partial^2 e_1/\partial v_1 \partial s_1, \qquad \partial(f_1,f_2)/\partial(p,v_1) = -\partial^2 e_1/\partial s_1 \partial v_1,$$

where Δ_1 is the hessian of the function $e_1(v_1, s_1)$. When the thermodynamic equilibrium of the one-phase state of the body is stable or metastable, and hence realizable at all, conditions of stability,

$$\partial^2 e_1/\partial v_1^2 > 0, \quad \partial^2 e_1/\partial s_1^2 > 0, \quad \Delta_1 > 0,$$

must be satisfied. It follows then that, in the neighborhood of any one-phase state of equilibrium,

$$\begin{array}{l} p \text{ and } \theta \text{ are continuous functions of } v_1, s_1, \\ p \text{ " } s_1 \text{ " " " " } v_1, \theta, \\ v_1 \text{ " } \theta \text{ " " " " } p, s_1, \\ v_1 \text{ " } s_1 \text{ " " " " } p, \theta\ ; \end{array} \tag{8}$$

and that

$$\begin{array}{l} \theta \text{ and} s_1 \text{ are continuous functions of } p, v_1, \\ p \text{ and } v_1 \text{ are continuous functions of } \theta, s_1, \end{array} \tag{9}$$

in the neighborhood of any point of any subregion throughout which $\partial^2 e_1/\partial v_1 \partial s_1$ differs from zero. The first result, that p and θ are functions of v_1, s_1, is of course initially given by the conditions of equilibrium themselves.

To interpret the condition $\partial^2 e_1/\partial v_1 \partial s_1 = 0$, we may differentiate the conditions (7). Writing e_{11}, e_{12}, e_{22} for the second derivatives of $e_1(v_1, s_1)$,

$$-dp = e_{11}dv_1 + e_{12}ds_1, \quad d\theta = e_{12}dv_1 + e_{22}ds_1,$$

and eliminating ds_1,

$$\Delta_1 dv_1 = -e_{22}dp - e_{12}d\theta,$$

whence follows

$$(\partial v_1/\partial \theta)_p = -e_{12}/\Delta_1.$$

Since $\Delta_1 > 0$, we have that the condition $e_{12} = 0$ is the equation of the locus representing states of maximum or minimum density at constant pressure. Such a locus exists in the field of states of thermodynamic equilibrium of bodies of liquid water.

Conclusions concerning single-valuedness of the functions (8) and (9) can be deduced from the theorem that any two states (p, θ, v_1, s_1) and ($p+\delta p$, $\theta+\delta\theta$, $v_1+\delta v_1$, $s_1+\delta s_1$) of a given region of one-phase states are in stable equilibrium only when

$$-\delta p \delta v_1 + \delta\theta \delta s_1 > 0. \tag{10}$$

Thus, for given values of v_1 and s_1, the function p of v_1, s_1 cannot have two values differing by $\delta p \neq 0$, since the values $\delta v_1 = \delta s_1 = 0$ and $\delta p \neq 0$ would not satisfy (10). In this manner we find that the eight functions (8) are single-valued when the equilibrium is stable. On the other hand it is not excluded that, for given values of p and v_1, the function θ of p_1, v_1 can have two values differing by $\delta\theta \neq 0$. For the values $\delta p = \delta v_1 = 0$ and $\delta\theta \neq 0$ would reduce (10) to $\delta\theta\delta s_1 > 0$, which merely requires that corresponding changes of θ and s_1 at unchanged p, v_1 must have

the same sign. In this manner we find that the four functions (9) are not necessarily single-valued. They are, in fact, multiple-valued for certain ranges of the independent variables in the field of states of a body of liquid water.

SUMMARY: *Let p be the pressure, θ the absolute temperature, v_1 the specific volume, and s_1 the specific entropy, of any one-component body in any region of realizable one-phase states of thermodynamic equilibrium. Any two of these variables are continuous functions of the other two. The eight functions of the several pairs of "work-heat variables"*

$$v_1, s_1, \quad v_1, \theta, \quad p, s_1, \quad p, \theta,$$

are single-valued; but the four functions of the "work variables" p, v_1 and the "heat variables" θ, s_1 may be multiple-valued.

The Case $n = 2$

When expressed by means of (6), the conditions (3), for $n = 2$, are:

$$-p = \frac{\partial e_1}{\partial v_1} = \frac{\partial e_2}{\partial v_2}, \quad \theta = \frac{\partial e_1}{\partial s_1} = \frac{\partial e_2}{\partial s_2},$$

$$e_1 - \frac{\partial e_1}{\partial v_1} v_1 - \frac{\partial e_1}{\partial s_1} s_1 = e_2 - \frac{\partial e_2}{\partial v_2} v_2 - \frac{\partial e_2}{\partial s_2} s_2,$$

which shall be written:

$$f_1 = p + \frac{\partial e_1}{\partial v_1} = 0, \qquad f_2 = p + \frac{\partial e_2}{\partial v_2} = 0,$$

$$f_3 = \theta - \frac{\partial e_1}{\partial s_1} = 0, \qquad f_4 = \theta - \frac{\partial e_2}{\partial s_2} = 0,$$

$$f_5 = e_1 - (\partial e_1/\partial v_1) v_1 - (\partial e_1/\partial s_1) s_1 - e_2 + (\partial e_2/\partial v_2) v_2 + (\partial e_2/\partial s_2) s_2 = 0.$$

When the body is in a two-phase state of thermodynamic equilibrium, these equations are satisfied by a set of values of the variables p, θ, v_1, s_1, v_2, s_2. And, in the neighborhood of this set of values, the five f_i are continuous and possess continuous first derivatives with regard to all their variables. Now, to determine whether θ, v_1, s_1, v_2, s_2, in this neighborhood, are single-valued continuous functions of the pressure p taken independent, we require to determine whether the jacobian of the five f_i with regard to these variables is non-vanishing at the point. This jacobian, and the corresponding jacobians when θ, v_1, s_1, v_2, s_2 are successively taken independent, are as follows:

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(\theta,v_1,s_1,v_2,s_2)} = -\Delta_1\Delta_2(s_2 - s_1), \tag{11}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,v_1,s_1,v_2,s_2)} = +\Delta_1\Delta_2(v_2 - v_1), \tag{12}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,\theta,s_1,v_2,s_2)} = +\Delta_1\Delta_2\psi_1, \tag{13}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,\theta,v_1,v_2,s_2)} = +\Delta_1\Delta_2\phi_1, \tag{14}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,\theta,v_1,s_1,s_2)} = +\Delta_1\Delta_2\psi_2, \tag{15}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,\theta,v_1,s_1,v_2)} = +\Delta_1\Delta_2\phi_2, \tag{16}$$

where Δ_i is the hessian of the function $e_i(v_i, s_i)$, and ϕ_i, ψ_i are the expressions'

$$\phi_i = \frac{1}{\Delta_i}\left(\frac{\partial^2 e_i}{\partial v_i^2}(v_2 - v_1) + \frac{\partial^2 e_i}{\partial v_i \partial s_i}(s_2 - s_1)\right), \qquad i = 1,2,$$

$$\psi_i = \frac{1}{\Delta_i}\left(\frac{\partial^2 e_i}{\partial v_i \partial s_i}(v_2 - v_1) + \frac{\partial^2 e_i}{\partial s_i^2}(s_2 - s_1)\right), \qquad i = 1,2.$$

When the thermodynamic equilibrium of two coexisting phases is stable or metastable, and hence realizable at all, conditions of stability $\Delta_1 > 0$ and $\Delta_2 > 0$ are satisfied. If we now assume, from physical experience, that coexisting phases have distinct specific volumes and specific entropies, and thus that an isothermal transformation of either phase into the other always involves a change of volume and an absorption or development of heat, we have

$$v_2 - v_1 \neq 0, \quad s_2 - s_1 \neq 0. \tag{17}$$

At a "critical point" it is indeed true that $v_2 = v_1$ and $s_2 = s_1$; but a critical state is not a two-phase state.

Because of these data it follows from (11) and (12) that, in the neighborhood of any two-phase state of equilibrium, the quantities θ, v_1, s_1, v_2, s_2, are continuous functions of p, and that the quantities p, v_1, s_1, v_2, s_2 are continuous functions of θ. To formulate the first derivatives of these functions, we divide each of the equations (12) to (16) by (11), and divide each of (11) and (13) to (16) by (12), member by member, and reduce by the theorem on the jacobians of implicit functions, thereby obtaining:

$$(18)\qquad \begin{aligned} \frac{d\theta}{dp} &= \frac{v_2 - v_1}{s_2 - s_1}, & \frac{dv_i}{dp} &= -\frac{\psi_i}{s_2 - s_1}, & \frac{ds_i}{dp} &= \frac{\phi_i}{s_2 - s_1}, \\ \frac{dp}{d\theta} &= \frac{s_2 - s_1}{v_2 - v_1}, & \frac{dv_i}{d\theta} &= -\frac{\psi_i}{v_2 - v_1}, & \frac{ds_i}{d\theta} &= \frac{\phi_i}{v_2 - v_1}. \end{aligned}$$

The equation for $dp/d\theta$ is the "Clapeyron-Clausius equation."

The dependence of p and θ on each other is represented in the p, θ plane by curve segments which intersect in points representing regions of three-phase states of equilibrium. By (17) and the Clapeyron-Clausius equation, none of these segments can have a horizontal or a vertical tangent. On each segment, therefore, p and θ are single-valued functions of each other.

A similar conclusion concerning v_i, s_i, regarded as functions of p or of θ, can be drawn. Within a given field of two-phase states of stable equilibrium take any two states having equal pressures, and hence equal temperatures. Let the quantity v_1 have the respective distinct values v_1 and $v_1+\delta v_1$ in the two states. Then the states of the first phase are states of the field of stable one-phase states for this phase, and hence are subject to the criterion of stability (10). But this inequality is not satisfied, since $\delta p = \delta\theta = 0$. Hence the two states of the first phase are not distinct—a transformation of either into the other is not a physical change of state. The quantity v_1 is therefore a *single-valued* function of p or of θ. If for v_1 we read s_1 or v_2 or s_2, the reasoning is the same. So we conclude that, along any continuous segments of the curves representing these quantities as functions of p or of θ, these functions are single-valued.

There remain for consideration the equations (13) to (16), where we require to examine the possible vanishing of the factors ϕ_i, ψ_i. By (18), $\psi_i = 0$ when and only when dv_i/dp and $dv_i/d\theta$ vanish—when the density of the ith phase is a maximum or minimum. A minimum occurs, for example, at the temperature of maximum density of liquid water which is "saturated" with respect to a phase of water vapor. When this example is represented in the V, p plane, or in the V, θ plane, or in the V, S plane, it becomes obvious that p, θ, s_1, v_2, s_2 are two-valued functions of the specific volume v_1 of the liquid, for a range of values of this variable. When the density of the ith phase is neither a maximum nor a minimum the first members of (13) and (15) differ from zero, wherefore p, θ, s_i, and the specific volume and entropy of the other phase are continuous functions of v_i. At a point of maximum or minimum v_i two branches of each function reach a common value.

Again, by (18) $\phi_i = 0$ when and only when ds_i/dp and $ds_i/d\theta$ vanish—when the specific entropy of the ith phase is a maximum or a minimum. Such a maximum often occurs, and even minima are observed. When either is represented in the S, p plane, in the S, θ plane, or in the S,V plane, it becomes obvious that p, θ, v_1, v_2, s_2 are two-valued, and sometimes even three-valued.

functions of the specific entropy s_1, for a range of values of this variable. When s_i is neither a maximum nor a minimum the first members of (14) and (16) differ from zero, wherefore p, θ, v_i, and the specific volume and entropy of the other phase are continuous functions of s_i. At a point of maximum or minimum s_i, two branches of each function reach a common value.

When the first derivatives of p, θ, v_i, s_i with regard to one another are formulated, by applying the theorem on the jacobians of implicit functions to the equations (11) to (16), they are found to form the set of equations:

$$\frac{dp}{s_2 - s_1} = \frac{d\theta}{v_2 - v_1} = -\frac{dv_1}{\psi_1} = -\frac{dv_2}{\psi_2} = \frac{ds_1}{\phi_1} = \frac{ds_2}{\phi_2}.$$

SUMMARY: *Let p be the pressure, θ the absolute temperature, and v_i, s_i the specific volumes and specific entropies of the phases, of any one-component body in any region of realizable two-phase states of thermodynamic equilibrium. Any five of these variables are continuous functions of the remaining variable. The five functions of p and the five functions of θ are single-valued; but the five functions of each v_i and of each s_i may be multiple-valued.*

The Case $n = 3$

When expressed by means of (6), the conditions (3) for $n = 3$ are:

$$f_1 = p + (\partial e_1/\partial v_1) = 0, \qquad f_5 = \theta - (\partial e_2/\partial s_2) = 0,$$
$$f_2 = p + (\partial e_2/\partial v_2) = 0, \qquad f_6 = \theta - (\partial e_3/\partial s_3) = 0,$$
$$f_3 = p + (\partial e_3/\partial v_3) = 0, \qquad f_7 = h_1 - h_2 = 0,$$
$$f_4 = \theta - (\partial e_1/\partial s_1) = 0, \qquad f_8 = h_1 - h_3 = 0,$$

where

$$h_i = e_i - (\partial e_i/\partial v_i)v_i - (\partial e_i/\partial s_i)s_i, \qquad i = 1,2,3.$$

To determine whether the eight conditions are independent, we form the jacobian of the eight functions f_i, obtaining

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8)}{\partial(p,\theta,v_1,s_1,v_2,s_2,v_3,s_3)} = -\Delta_1\Delta_2\Delta_3 \begin{vmatrix} v_1 & v_2 & v_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{vmatrix}, \tag{19}$$

where, as before, Δ_i denotes the hessian of the function $e_i(v_i, s_i)$.

Let us consider the determinant of order three in (19). From the equations for the volume V and the entropy S of the body of mass c,

$$\begin{aligned} M_1v_1 + M_2v_2 + M_3v_3 &= V, \\ M_1s_1 + M_2s_2 + M_3s_3 &= S, \\ M_1 + M_2 + M_3 &= c, \end{aligned} \tag{20}$$

we find, for example,

$$\begin{vmatrix} v_1 & v_2 & v_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{vmatrix} M_1 = \begin{vmatrix} V & v_2 & v_3 \\ S & s_2 & s_3 \\ c & 1 & 1 \end{vmatrix}.$$

If the coefficient of M_1 should vanish, the second member of this equation would vanish also, and the equations (20) would be satisfied by an infinite number of values of M_1, M_2, M_3, while v_i, s_i, V, S remained constant. But this is impossible, since the state of equilibrium of the body cannot be altered at constant V, S. So the coefficient of M_1 is different from zero. Further, by conditions of stability, the hessians Δ_1, Δ_2, Δ_3 in (19) are positive for realizable states of equilibrium. It follows that the eight variables are connected by eight independent relations, and hence are constants.

THE SCIENCE OF CALCULATION BY THE BOARD

By BIBHUTIBHUSAN DATTA, University of Calcutta.

In a previous issue of this Monthly[1] it has been observed by Professors David Eugene Smith and Salih Mourad that the word *takht* appears in the titles of several treatises on arithmetic by the eastern Moslem scholars of the tenth century A.D. and later. They have shown that the Arabic word *takht* is derived from the Persian word *takhta* and that it means "the board" or "the wood," not "the method" or "the table" as supposed by previous writers. The so-called ghobar-numerals (*Haruf al-ghobar*) of the western Arabs are ordinarily believed to have derived their name from the ancient practice of writing them on a wooden board covered with a thin layer of dust or sand (the Arabic *ghobar*). This writing of the learned authors reminds me of a term in the Hindu mathematics, which has an identical significance with *takht*, but which occupies a position of far greater importance. A comparative study of the two terms is bound to disclose how much deeper was the relation between the cultures of the two neighboring civilizations, so far at least as mathematics was concerned. It forms, indeed, a very interesting and instructive study from the point of view of the historian of mathematics. I propose in this paper to lay before the readers of this Monthly a brief statement of this relation.

The term in view is *pāṭi* or *pāṭī*. The modern Sanskrit word *pāṭi* is of doubtful origin. It does not appear to be purely Sanskrit, for it is not found in

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[1] Vol. 33 (1927), pp. 258–300.

the ancient writings and kindred literatures of India,[1] nor does it occur in the earlier Sanskrit lexicons such as the *Nirukta* and the *Amarakoṣa*. We miss it even in the *Abhidhānaratnamālā* of Halāyudha in the eleventh century. According to Gaṇeśa (1545), the eminent mathematician and commentator of Bhāskarācārya, the term *pāṭi* means "the express order of sequence of addition, etc."; and hence it denotes "arithmetic." Another commentator, Munīśvara (born 1603) takes it as having the same significance as *paripāṭi*.[2] This latter is a pure Sanskrit word and is contained in the *Amarakoṣa*. It means "succession," "method" or "the arrangement." These interpretations of the word *pāṭi* have found place in all the later Sanskrit lexicons, but the exact derivation of the word is not found in any dictionary. In the popular tongues of India, the word *pāṭi* means "the board," usually "the writing board." Hence it seems originally to have been a corrupt form of the pure Sanskrit word *paṭṭa*, which means "the slab," "the tablet,"[3] and which frequently occurs in the earlier Indian literatures. The still more ancient and common word for the board was *phalaka*.[4]

The use of the writing board has been common in India since at least five centuries before the Christian era. There is mention of it in the Buddhist *Vinaya Piṭaka*,[5] the *Jātaka*,[6] and the *Lalita-vistara*,[7] and in the other Sanskrit and kindred literatures of India.[8] The wooden writing board is still found to be in use in village elementary schools in some parts of India, though in other parts it is being gradually replaced by the modern slate boards. The students used to write on the board in early days, as at present, with the *varṇaka* (literally coloring rod, meaning the style or the pen), or the *pāṇḍu-lekha* or the *śveta-varṇi*, that is, with a piece of yellow or white soapstone.[9] Among the astronomers the more common practice seems to have been to cover the board with a thin layer of dust and then to do computing with a style. It is probably for this reason that the astronomical computation is called the *Dhūli-karma*

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[1] The word *pāṭi* occurs in the *Vimāna-vattu* commentary of the Buddhists but its significance is not clear. Compare the *Pāli-English Dictionary*, by T. W. Rhys Davids and William Stede.

[2] See the commentary on Bhāskara's *Līlāvatī* by Ganeśa and by Munīśvara; also H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhāscara*, London, 1817, p. 1 ff.

[3] The word *paripāṭi* itself was possibly derived originally from the word *pāṭi* with the addition of the prefix *pari*.

[4] The word *phalaka*, meaning "the board," "the plank," "the leaf," occurs in the *Brāhmanas* and the *Grhya-* and *Srauta-sūtras* and also in the post-Vedic literatures of India. The word *paṭṭa* occurs in the *Mahābhārata*, *Harivamsa*, and the Buddhist *Nikāyas* and *Jātakas*.

[5] *Vinaya Piṭaka*.

[6] *Jātaka*, Fausboll, I, p. 451; Robert Chalmer's translation of the Jatakas, I, p. 275.

[7] *Lalitavistara*, ed. Rajendralal Mitra, Calcutta, p. 143.

[8] Compare Bühler's *Indian Palaeography*, English translation by J. F. Fleet, p. 5 and §37, C.

[9] At present, chalk is also used.

(literally the "dust-work") by Brahmagupta and Bhāskara.[1] It is strange that the practice continued to be in vogue among the Hindu astronomers in later times even when improved and better materials for writing were known and used by the people.[2] The celebrated Arabian traveller Al-Bîrunî (*c.* 1030) informs us that the practice of writing on the dust in his time was prevalent in Kashmir.[3]

The above practice of doing numerical computations on the dust-board must have been carried to Arabia by the Hindu astronomers or others; and I presume that the Arabic terms, *Ḥisāb al-ghobar* for Indian arithmetic and *Haruf al-ghobar* for the Indian numeral figures must have originated in that connection.[4] In fact the term *Ḥisāb al-ghobar* is nothing but the Arabic version of the Sanskrit term *Dhūri-karma*.

In later times the term *pāṭi* came to be very closely associated with the science of mathematics; so that arithmetic (including geometry) is called *Pāṭi-gaṇita*, which literally means "the science of calculation on the *pāṭi*." This is exactly the title of a treatise on arithmetic by Sinên ibn al-Fath (c. 940), a native of Harran, *Ilm hisāb al-takht* ("The science of calculation of the *takht*").[5] The Arabic word *hisāb* is equivalent to the Sanskrit word *gaṇita*.

In India, arithmetic was some times called the *Pāṭi-śāstra* ("The book of knowledge of the *pāṭi*" or "The book of the *pāṭi*") or briefly *pāṭi*.[6] Similarly, we get in Arabia *Kitab al-takht* ("The book of the *takht*") as the title of a work by ar-Râzî. It has been stated before that the expression *kitâb al-takht* or *hisâb al-takht* occurs in the title of many Arabic works on elementary mathematics.

According to the celebrated Hindu mathematician Bhāskarācārya (born 1114), there are two principal branches of the *Gaṇita*, namely the *Paṭi-gaṇita* and the *Vīja-gaṇita*. The former branch corresponds to modern arithmetic (including geometry and mensuration) and the latter to algebra. They are also called respectively the *Vyakta-gaṇita* ("The science of calculation with the known") and the *Avyakta-gaṇita* ("The science of calculation with the unknown").[7]

The title *Pāṭi-gaṇita* was also given to some of the earlier Hindu works on

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[1] *Brāhma-sphuṭa-siddhānta*, x. 62, 66, 67. *Siddhānta-siromani*, *Yantrādhyāya*, verse 24 (*vāsanā*). I am indebted to Mr. Probodhchandra Sengupta for this information.

[2] It may be noted that the astrologers in Bengal, even now, begin nativity calculations on a wooden board covered with dust.

[3] *Alberuni's India*, English translation by E. C. Sachau, London, 1878, vol. 1, p. 174.

[4] A commentary on the *Sēfer Yesīrāh*, composed about 950, speaks of the Indian arithmetic known under the name of *ghobar* calculation. (Smith and Karpinski, *Hindu-Arabic Numerals*, Boston, 1911, pp. 67–8).

[5] The *Fihrist*, pp. 37, 70; Arabic text, p. 281.

[6] *Mahāsiddhānta*, i. 1, XV. I. Also compare the colophon of Śrīdhara's *Triśatikā*.

[7] *Siddhanta-śiromani*.

arithmetic. Thus we have *Pāṭi gaṇita-sāra* ("The essence of calculation of the board") as the title of the arithmetical treatise of Śrīdhara (c. 750). This work is more commonly known as *Triśatikā* from the fact that it contains 300 couplets. The author states in the opening verse of the book that it is a compendium of his larger work on the *Pāṭi-gaṇita*.[1] This latter title was probably given to the twelvth chapter of Brahmagupta's larger astronomical treatise, *Brāhma-sphuṭa-siddhānta*.[2] We have also the *Pāṭi-gaṇita* by Âryabhaṭa the Younger (c. 950),[3] the *Pāṭi-gaṇita-kaumudi* by Nārāyaṇa (1357), and the *Pāṭi sāra* of Munīśvara. In modern times, in Sanskrit and in some of the important vernaculars of India, *pāṭi-gaṇita* is the common name for a treatise on arithmetic.[4]

It is certainly rather strange that such a term as *pāṭi* or "the board" came to be so closely connected and almost identified with the Hindu arithmetic. Sankar Balkrisna Dikshit thinks that the name *pāṭi-gaṇita* for arithmetic has its origin in the practice of the Hindu astrologers in using a wooden plank for calculation.[5] He has not clearly explained, however, how far and in what important respect the board is related to arithmetic; whether its presence in the Hindu name for arithmetic refers simply to the ancient and more common use of the board as the writing material for the arithmetical operations; or whether it goes still deeper and implies the use of the board as the "abacus," or an instrument of calculation which was so essential for expressing the place value before the invention of the zero. Bayley[6] and Fleet[7] are of the latter opinion. They suspect that the origin of the term *paṭi* in the name of the Hindu arithmetic lies in the use of the board as an abacus in India at some distant date. That this conjecture is without foundation is evident for the following reasons: First, it is very doubtful if the abacus, in any form, ever existed in India. No such trace of it, whether literary or palaeographic, has as

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[1] Compare Śrīdhara's text *pāṭyā ganitasya* with the Arabic *hisāb al-takht*.

[2] We say "probably" because, as has been observed by Sudhakara Dvivedi (*Brāhma-sphuṭa-siddhānta*, Benares, 1902), the full title *Pāṭi-ganitādhyāya* ("the chapter on the *Pāṭi-ganita*") appears in some of the available manuscripts, while in others the word *pāṭi* is dropped.

[3] This is the title of the 15th chapter of the *Mahā siddhānta*.

[4] In southern and western parts of India arithmetic is called *anka-ganita*, while in northern and eastern India it is called *pāṭi-ganita*.

[5] *Indian Antiquary*, vol. 20, 1891, p. 54, footnote 7. R. C. Temple tells of a similar custom being prevalent among the Burmans even in the nineteenth century. "The Burman does his calculations either on the ground in the dust or on black *parabaik*. *Parabaik* is a thick course indigenous paper with a smooth greasy surface on which characters are written with a soft soapstone style." (*Ibid.* p. 54.) As stated in the first footnote on p. 522, the astrologers in Bengal, even now, begin nativity calculations on a wooden board covered with dust.

[6] Journal of the Royal Asiatic Society, N. S., vol. 14 (1882), p 369, footnote 3.

[7] J. F. Fleet, *The use of the abacus in India*, Journal of the Royal Asiatic Society (1911), p. 519.

yet been discovered.[1] Second, the term *pāṭi-gaṇita* does not appear so far as is known, before the seventh century A.D., but the abacus, if it had ever existed in India, must have disappeared long before that time, for we now know definitely that the place value and the zero existed in India even before 200 B.C.[2] Third, if the board abacus, before its complete disappearance from the country, had in reality exerted any influence on the coining of a name for Hindu arithmetic, the earliest name for the latter would have been *phalàka-gaṇita* or *paṭṭa-gaṇita*; for as we have aready pointed out, the oldest Sanskrit term for the board was *phalaka* or *paṭṭa*, not *pāṭi*. The total absence of a term like this from the earliest Indian literatures is highly significant.[3]

Professor Jogeshchandra Roy is of opinion that the term *pāṭi* as occurring in the name of the Hindu arithmetic has no connection whatsoever with the board abacus. It is used only to distinguish one branch of arithmetic from the other. He says that a branch, rather the higher branch of the science of the *gaṇita*, came to be called by the name *pāṭi-gaṇita* on account of the indispensable need of using the *pāṭi*, as merely the writing material, for performing all the necessary operations in it. It was thus separated from the lower or (more) elementary branch in which all the operations could be carried on mentally, without the aid of any writing material or of any other external agency.[4] This seems to be the most correct view of the origin of the name *pāṭi-gaṇita*.[5] In modern times in India it is customary to distinguish between the *pāṭi-gaṇita* and the *mānasānka* (or "the mental arithmetic").

This practice of differentiating between the elementary and the higher classes of arithmetic seems to be a very ancient one. In the earlier Buddhist

[1] It has been pointed out elsewhere (this Monthly, vol. 33 (1926), p. 450, footnote 5) that the instance quoted by Fleet in support of the use of the abacus in India implies something quite different.

[2] Bibhutibhusan Datta, *Early literary evidence of the use of the zero in India*, this Monthly, vol. 33 (1926), pp. 449–454.

[3] In the earlier literatures of India, we occasionally meet with such terms as *lipi-phalaka* (writing board), *citra-phalaka* (drawing or painting board), *śāri-phalaka* (gaming board), *sphatika-phalaka* (slab at the base of the pedestal), etc. Some of these terms, especially the compounds with the parts of the body (*aṁśa-*, *Jánuśroṇi-*, etc.) are very ancient and occur in the *Brāhmanas*. Similarly, we have *citra-paṭṭa*, *dvāra-paṭṭa*, *ayo-paṭṭa*, *tāmra-paṭṭa*, etc. But never do we meet with any such term as *gananā-phalaka* or *gananā-paṭṭa* (counting board). There would have remained a trace of the counting board in the literatures, if it had ever existed in India.

[4] This opinion was given by Professor Roy in a personal letter to the writer in approving of and elaborating upon a similar suggestion of his own.

[5] This hypothesis is supported by the following remark of Keśava (c. 1450). In stating the object of writing a handbook of astronomy, *Grahakautuka*, Keśava observes that there are indeed many manuals which give the position of the stars, more or less accurately; but calculations according to them can not be done easily without the help of the board (*paṭṭa*). His object was to write a book which would help to calculate the position of the stars easily without using a *paṭṭa*. What Keśava undoubtedly meant was that the calculations required for his handbook were to be so simple that they could be done mentally. (Cf. *Ganaka Tarangini*, Sudhakara Dvivedi, Benares, 1892, p. 58). My attention was drawn to this passage by Mr. Gurugovinda Chakrabarti.

literature we find separate mention of three classes, viz., *mudrā*, *gaṇanā*, and *saṅkhyāna*. Following a Simhalese commentator, Rhys Davids takes *mudrā* as meaning "counting on the fingers."[1] From the Tibetan source, Schiefner gives it the German title "Handrechnen," while Franke translated it as "Finger Rechnen." *Gaṇanā* is rendered by Rhys Davids as "counting without using the fingers," meaning "mental arithmetic, pure and simple." All these meanings are quite in agreement with the interpretations of the eminent Buddhist commentator Buddhaghosa, although Rhys Davids is not sufficiently clear and precise, perhaps due to his failure to grasp fully the true significane of *saṅkhyāna*, taking it to mean "summing up large totals." It really means arithmetic in general—literally "the science of numbers." What was undoubtedly intended by the term *saṅkhyāna* is the upper class of arithmetic in contradistinction to the other two classes mentioned. One of the earliest enumerations of these three classes occurs in the *Digha Nikāya*,[2] and it is also found in the *Vinaya Piṭaka*,[3] *Divyāvadāna*[4] and *Milindapanho*.[5] Thus as early as the fifth or the sixth century before the Christian era the Indians were accustomed to distinguish between three classes of arithmetic; (1) mental arithmetic, pure and simple; (2) arithmetic with the use of the fingers; and (3) higher arithmetic in general. From the way in which this classification was made it is evident that some kind of writing material was essential for this third class of arithmetic. The term *pāṭi-gaṇita* must have been originated, in a later age, as a specific and distinctive name for it.

It is also noteworthy that the above classification of arithmetic is met with in Buddhist writings, not in other Indian literatures. Even in the former the different classes of arithmetic are not always distinguished, arithmetic in general being referred to as *gaṇanā*, *gaṇita*, or *saṅkhyāna*.[6] For example, in the *Vinaya Piṭaka*,[7] as also in the Hathigumpa inscriptions,[8] there is only general mention of the *gaṇanā* (arithmetic) together with *lekhā* (writing) and *rupa* (drawing)[9] as a subject of elementary study for a beginner. Of course in

[1] T. W. Rhys Davids, *Dialogues of the Buddha*, vol. 1, London, 1899, p. 21; *Milinda-pañho*, English translation by Rhys Davids, Oxford, 1890, p. 91 footnote. The Simhalese commentator is very explicit; according to him *mudrā* denotes "the finger-ring art, so-called from seizing on the joints of the fingers, and using them as signs."

[2] I. 51.

[3] IV. 7.

[4] *Divyāvadāna*, edited by E. B. Cowell and R. A. Neil, Cambridge, 1886, pp. 3, 26, 58.

[5] *Milinda pañho*, loc. cit., p. 91.

[6] For instance see *Jātaka*, I. 29; *Visuddhi mägga*, p. 278; *Sutta-Nipāta*, verse 677, *Milindapañho*, p. 79.

[7] I, 77; IV, 129.

[8] Hathigumpa and three other inscriptions, ed. Bhagawanlal Indraji, p. 22.

[9] *Rupa* includes painting as well as drawing. For amongst the illustrations of *rupa* enumerated in the Buddhist *Dhammasangini*, 617 (ed. by Edward Müller, London, 1885, p. 139, and English transla-

some of those instances, *ganaṇā* may be taken to mean mental arithmetic—the class (1) mentioned before; but in others it certainly denotes arithmetic in general. Similarly, in the Kauṭilaya's *Arthaśāstra*,[1] the word *saṅkhyāna* has been used instead of *ganaṇā*. The Jaina literatures speak of *Saṅkhyāna* (arithmetic) and *Jyotiṣa* (astronomy) as two of the *Brahmanical sciences*, as early as 300 B.C.[2] In the *Amarakoṣa* (c. 450 A.D.), the *gaṇita* and the *saṅkhyāta* are used synonymously. In several Sanskrit works an expert arithmetician is spoken of as a *saṅkhyā-gaṇita-tattvajna*.[3]

From what has been stated above, it follows that the name *paṭi-gaṇita* originated probably in a non-Brahmanical literature of India, a vernacular of northern India. This accounts to a certain extent for the presence in it of a word having the stamp of non-Sanskrit influence. The term *pāṭi-gaṇita* as well as the word *pāṭi*, however, entered into the Sanskrit literature in a later age, probably about the seventh century A.D.

Turning now to Arabia, we find strikingly close similarity in the title of several treatises on arithmetic belonging to the two countries. Two instances of a literal translation have been noticed before. There are others in which the titles of the Arabic treatises are directly adopted from the Hindu source. The title of Al-Kālwādani's arithmetical treatise *Kitāb al-takht fi'l hisāb al-Hindi* is obviously the Arabic version for *Hindu pāṭi-gaṇita* and Al-Antāki's *Kitāb al-takht al-kabir fi'l hisāb ab-Hindi* is nothing but the Arabic for "The great Hindu pāṭi-ganita." Suter has noticed eight Arabic treatises in the title of which occurs the word *takht* and he has always connected it with Hindu arithmetic.[4] This is very significant, and it all goes to show that the source of the Arabic term *Hisāb al-takht* for arithmetic must be traced to India.[5] The Arabs seem to have even followed the old Hindu practice of dividing arithmetic into the *pāṭi-gaṇita* and the *mudrā* (*-gaṇita*). This is shown by the titles of the works of Al-Antāki; for besides the one on *Hisāb al-takht* we find a title of another work by the same writer, *Kitāb al-hisāb bila takht bal bi'l yad* ("The book on arithmetic with the hand without the board)."

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tion by Caroline A. F. Rhys Davids, *Buddhist Psychology*, London, 1900, p. 123) there are colors as well as geometric figures. The date of this book is 350 B.C. or earlier.

[1] Kauṭilya's *Arthaśāstra*, ed. and English Translation by R. Shamashastri, i. 5, 2.

[2] The *Kalpasūtra of Bhadrabāhu*, ed. by H. Jacobi, Leipzig, 1897, *Bhagabati-sūtrā*, Bombay, 1918, p. 112.

[3] For instance *Mahābhārata*, i. 293; *Mrcchaṭika*, i. 15; *Kathāsaria-sāgara*, vi. 32; *Brihat Sanilutā* xi. 2.

[4] H. Suter, *Die Mathematiker und Astronomer der Araber und ihre Werke*, Leipzig, 1900, pp. 31, 64, 66, 74, 149, 160.

[5] Cf. Dr. Solomon Gandz, *On three interesting terms relating to area*, this Monthly, vol. 34 (1927), pp. 80–86. Dr. Gandz has shown that all those three terms entered into Arabia from India. He has further remarked, "Naturally we turn to Hindu literature, the source of so many ideas of Al-Khowârîzmî . . ." (p. 83).

Nearly five months after the foregoing portion of the paper left my hands, there came through the courtesy of the author, a reprint of a paper by Dr. Solomon Gandz of New York, *Did the Arabs know the abacus?*, which had appeared in a previous issue of this Monthly.[1] As it deals primarily with the problem which forms the subject matter of the present study, it is proper to refer to it here. The learned writer, after making a retrospective study of what various other scholars have stated before him about the existence of the abacus among the Arabs, discusses anew, as an introduction to his own theory on the subject, the question of how the term *takht* or *Kitāb al-takht* came to be identified with the name of the science of arithmetic among the eastern Arabs. This question was raised and discussed previously by Professors Smith and Mourad, and an exactly similar question has been raised by Indologists about the identification of the term *pāṭi* with the name of arithmetic among the Hindus. It has been said very rightly that as all treatises of the eastern Arabs named *Kitāb al-takht* or *Hisāb al-takht* and those of the western Arabs named *Ilm al-ghobār* or *Hisāb al-ghobār* dealt with the Hindu arithmetic, any theory to be correct and reliable must have reference to this latter source. Smith and Mourad are of the opinion that the Hindu arithmetic was so named by the Arabs on account of the ancient practice of computations carried out on a wooden board sprinkled with a thin layer of dust, with the help of a stylus; that is, on the dust-board. Such was also the opinion of Smith and Karpinski[2] and of Weissenborn;[3] and a similar hypothesis, as already stated, was adumbrated by Bayley and Fleet as regards the origin and growth of the name *Pāṭīgaṇita*.

Gandz, however, does not think that the use of the dust-board as a crude writing device gave rise to the name of the science of arithmetic. "It can hardly be assumed," says he, "that a kind of a board tablet of the Arabic school is meant. There is nothing characteristic in such a board to give the name to the science of arithmetic" (p. 313). He therefore ventures to make the bold conjecture that the words *takht*, *ghobār*, and *turāb* are simply the Arabic terms for the translation of the Greek term *abakion* (abacus)—the Semitic *abaq*. He observes further that they need not necessarily refer to the dust-abacus alone, but the abacus in general, including the counter- and line-abacus. As regards the *raison d'être* of the Hindu arithmetic being called the abacus arithmetic, Gandz remarks: "Now we know the characteristic of the Hindu numerals was the place value and the zero. This was also the characteristic of the abacus. It would therefore be proper to call the Hindu arithmetic the abacus arith-

[1] Vol. 34 (1927), pp. 308–316. I am sorry that this interesting article escaped my notice when it first appeared.

[2] Smith-Karpinski, *The Hindu-Arabic Numerals*, p. 65.

[3] Quoted in extenso by Dr. Gandz.

metic" (p. 313). He admits however, some weakness in his hypothesis, based as it is on a certain interpretation of the words *takht*, *ghobār*, and *turāb*, which is altogether new and revolutionary and is moreover unknown to Arabic philology and mathematics. Nevertheless he maintains his position on the ground that his explanation is not only better and more reasonable than those usually adduced, but is also bound, in his opinion, to "fit more smoothly into the general history of mathematics and throw more light on the data and terms used by the Europeans." By way of illustration of his last-mentioned contention, he mentions the fact that the work of Leonardo Fibonacci (1202), which did so much to make known the Hindu arithmetic and which was dependent upon Arabic sources, bore the name *Liber Abaci*. He also calls attention to the further fact that during the middle ages in Europe the word abacus finally came to mean any kind of elementary arithmetic, and even books dealing with algorismus and the Hindu numerals bore the name *Liber Abaci* or *Libro d'abaco*.[1]

The theory which the present writer has here suggested with a view to solving what has hitherto remained a knotty problem of the history of mathematics can be briefly summarised thus: First, on the Arabian side: (1) the terms *Hisāb al-takht* and *Kitāb al-takht* of the eastern Arabs are nothing else than the Arabic version of the Sanskrit *Pāṭīgaṇita* or *Pāṭīsāstra*; (2) the *Hisāb al-ghobār* and other similar terms of the western Arabs are mere adaptations of the older Sanskrit term *Dhūli-karma*; Secondly, on the Hindu side: the terms *Pāṭīgaṇita* and *Dhūli-karma* originated and grew primarily as the name of a type of mathematics for which some kind of writing materials was found indispensable, in contradistinction to two other more elementary types of arithmetic, namely the mental arithmetic and the finger arithmetic. In the case of the first term the writing materials contemplated were the wooden board and a piece of soapstone; and in case of the other they were the stylus and a wooden board covered with a layer of fine sand or dust. In either case the board was never meant to be a means of indicating the place value and the

[1] The analogy of the word "abacus" lending its name to treatises of elementary arithmetic in general does not seem to be particularly happy and to the point. It is well known that, in the early middle ages, the arithmetics employing the Hindu numerals were called the *algorismus* (*algorismi*) whereas those employing the Roman numerals and in which the calculations were made by the counter-abacus were called the "abacus arithmetic" or simply the "abacus." This difference in nomenclature and the rivalry between the two schools continued for a long time. (D. E. Smith, *Rara Arithmetica*, pp. 6-7). If finally the "abacus" sometimes lent its name to books dealing even with the algorismus and the Hindu numerals, the algorismus at other times and exactly in the same way also lent its name to treatises employing the abacus. In fact these two words were loosely used, and a treatise on arithmetic in general could bear either name. (D. E. Smith, *History*, II, pp. 7, 10, 11). We have not so far met with any trace of a similar contest in nomenclature for Hindu arithmetic among the Arabs. This is significant. It should also be noted that the early Latin translation of the arithmetic of Al-Kowârîzmî was called *Liber Algorismi de numero Indorum* (The Book of Al-Khowârîzmî on Hindu Numbers.)

zero. This theory is more comprehensive than that of Gandz, since it embraces also the Hindu side which he has overlooked. Indeed Gandz's theory fails to explain the origin and source of the Hindu terms. Again, from the fact that the western Arabs adopted and continued to use a term which went practically into disuse in the land of its birth, there follows the irresistible conclusion that the Hindu arithmetic reached the Arabs of the West long before it reached the Arabs of the East. Hence the present theory is in perfect accord with and is a kind of corroboration of the ingenious theory of Woepcke relating to the differences between the Hindu numerals of the eastern and the western Arabs.[1]

THE USE OF THE DISCRIMINANT IN FINDING THE SINGULAR SOLUTION OF $f(x, y, p)=0$

By G. H. LIGHT, University of Colorado

The theory of singular solutions, as given by Cayley,[2] is founded upon the fact that the singular solution must satisfy the differential equation $f(x, y, p)=0$, since x, y, and p are the same for the singular solution and the general solution of $f(x, y, p)=0$ at all points on the singular solution. In order to separate the singular solution from the general solution of $f(x, y, p)=0$ we must examine more closely the behaviour of p. To do this, suppose $f(x, y, p)=0$ to be written in the form

$$(1) \qquad y = f_1(x, p) \quad \text{or} \quad (2) \quad x = f_2(y, p).$$

This will not change the singular solution. Hence the singular solution will still be present when $f(x, y, p)=0$ has been changed into either

$$\phi\left(x, p, \frac{dp}{dx}\right) = 0 \quad \text{or} \quad \psi\left(y, p, \frac{dp}{dy}\right) = 0$$

by differentiating (1) and (2), respectively; only its form will now be $r(x, p)=0$, $s(y, p)=0$, or $t(p)=0$—since it is a function of x and y only, say $h(x, y)=0$. Therefore dp/dx or dp/dy must represent the change that takes place in p along the general solution only, having nothing to do with the singular solution; in other words, when $f(x, y, p)=0$ is changed into

$$\phi\left(x, p, \frac{dp}{dx}\right) = 0 \quad \text{or} \quad \psi\left(y, p, \frac{dp}{dy}\right) = 0$$

by differentiating (1) and (2) with respect to x and y, respectively, then

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[1] *Journ. Asiat.* t. 1 (6), 1863; also compare Florian Cajori, *History*, pp. 100–101.

[2] Messenger of Mathematics, vol. 2 (1873), pp. 6–12, vol. 6 (1877), pp. 23–37.

$h(x, y)=0$ changes into $r(x, p)=0$, $s(y, p)=0$, or $t(p)=0$; and the left member of these equations must now appear as a factor of

$$\phi\left(x, p, \frac{dp}{dx}\right) \text{ or } \psi\left(y, p, \frac{dp}{dy}\right).$$

This means that $\phi(x, p, dp/dx)$ has at least two known factors and may be written in the form

$$\phi(x,p,dp/dx) \equiv \lambda_1 r(x,p)\phi_1(x,p,dp/dx), \tag{3}$$

where λ_1 contains the remaining factors of $\phi(x, p, dp/dx)$. Similarly,

$$\psi(y,p,dp/dy) \equiv \lambda_2\, s(y,p)\, \psi_1(y,p,dp/dy), \tag{4}$$

and λ_2 contains the remaining factors of $\psi(y, p, dp/dy)$. When $h(x, y)=0$ changes into $t(p)=0$, both (3) and (4) must have $t(p)$ as a factor.

Since the vanishing of the discriminant is the condition that the variable, in this case p, shall have at least two equal roots, the singular solution must be represented by an equation $h(x, y)=0$, where $h(x, y)$ is a factor of $B(x, y)$ and $B(x, y)=0$ is the discriminant equation of $f(x, y, p)=0$.

This is exactly what takes place in practice; for, when we change $f(x, y, p)=0$ into

$$(1) \qquad y = f_1(x,p) \quad \text{or} \quad (2) \quad x = f_2(y,p)$$

by solving $f(x, y, p)=0$ for y and x, respectively, and then eliminating y and x from (1) and (2) by differentiation, we at once get

$$\phi\left(x, p, \frac{dp}{dx}\right) = 0 \text{ or } \psi\left(y, p, \frac{dp}{dy}\right) = 0.$$

These decompose into (3) or (4), and the singular solution is in the form $r(x, p)$, $s(y, p)$, or $t(p)$, while the general solution[1] is obtained from

$$\phi_1\left(x, p, \frac{dp}{dx}\right) = 0 \text{ or } \psi_1\left(y, p, \frac{dp}{dy}\right) = 0.$$

The purpose of this paper is to show how the factor $r(x, p)$, $s(y, p)$, or $t(p)$ can be found before solving $f(x, y, p)=0$. In what follows, $f(x, y, p)=0$ is to be considered:

(a) as a polynomial in x, y, p,

(b) as not being factorable as to p,

(c) as having a singular solution.

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[1] The general solution is found by solving $\phi_1(x, p, dp/dx)=0$ or $\psi_1(y, p, dp/dy)=0$, obtaining $p=f_1(x, c)$ or $p=f_2(y, c)$ and using the p thus found to eliminate p from $f(x, y, p)=0$.

Consider the type of differential equations that can be written in the form

$$f(x, y, p) = 0, \quad p = dy/dx,$$

which are of degree n in p, $n \geqq 2$.

Let the p-discriminant of $f(x, y, p) = 0$ be written in the form

$$B(x, y) = Cf(x)g(y)h(x, y).$$

As far as this paper is concerned, the factors $f(x)$ and $g(y)$ are considered as being trivial, since they lead to the obvious values ∞ and 0, respectively, for p, where p is the value of p along the singular solution.

In order to find p for the singular solution, we proceed as follows. Let $h(x, y) = 0$. Then $p = -(h_x/h_y)$. We now make use of $h(x, y) = 0$ to express $p = -(h_x/h_y)$ as a polynomial in x and p, say $r(x, p) = 0$; or as a polynomial in y and p, say $s(y, p) = 0$; or as a polynomial in p, say $t(p) = 0$.

Let us take $r(x, p)^1 = 0$ as the form that $h(x, y) = 0$ assumes; when the p, obtained by solving $r(x, p) = 0$ for p, along with the y we get by solving $h(x, y) = 0$ for y, satisfy $f(x, y, p) = 0$, then this p becomes the p of the singular solution. We then solve $f(x, y, p) = 0$ for y, getting (1) $y = f_1(x, p)$, and we eliminate y from (1) by differentiating it with respect to x, thus obtaining $\phi(x, p, dp/dx) = 0$, in which the differential equation of the general solution is distinguished from the differential equation of the singular solution by the new variable dp/dx which cannot be present in $r(x, p)$ since the singular solution is a function of x and y only and since differentiating $h(x, y) = 0$ with respect to x can only introduce p into the new form, $r(x, p)$, for $h(x, y) = 0$.

Therefore, when $f(x, y, p) = 0$ is changed into $\phi(x, p, dp/dx) = 0$ by changing $f(x, y, p) = 0$ into (1) $y = f_1(x, p)$ and by eliminating y from (1) by differentiating it with respect to x, then $r(x, p)$ must appear as a factor $\phi(x, p, dp/dx)$; and another factor becomes $\phi_1(x, p, dp/dx)$, this factor leading to the general solution.

The same argument applies when $s(y, p) = 0$ is taken instead of $r(x, p) = 0$. We have only to interchange y and x in what has been said, obtaining $\psi(y, p, dp/dy) = 0$; and $s(y, p)$ must appear as a factor of $\psi(y, p, dp/dy)$ leaving $\psi_1(y, p, dp/dy) = 0$ to furnish the general solution. It sometimes happens that the polynomial in p is a function of p only, say $t(p)$; this occurs only when $h(x, y) = 0$ has a particular form, and does not affect the argument just given. When this occurs, $r(x, p) = s(y, p) = t(p)$, and the factor $t(p)$ will be present in both

$$\phi\left(x, p, \frac{dp}{dx}\right) \text{ and } \psi\left(y, p, \frac{dp}{dy}\right),$$

[1] This does not mean that we cannot take $s(y, p) = 0$ and arrive at the same general solution.

and we can use either

$$\phi_1\left(x, p, \frac{dp}{dx}\right) = 0 \text{ or } \psi_1\left(y, p, \frac{dp}{dy}\right) = 0$$

to obtain the general solution.

It remains to find what relation must hold between the coefficients and exponents of $h(x, y)=0$ in order to determine whether $r(x, p)$, $s(y, p)$, or $t(p)$ is to be the factor that occurs in $\phi(x, p, dp/dx)$ or $\psi(y, p, dp/dy)$. The three most common forms for $h(x, y)=0$ are considered but the method is general.

Case 1.

$$h(x,y) = ax^m + by^n = 0.$$

Here

(1) $$p = -ab^{-1} \cdot mn^{-1} \cdot x^{m-1}y^{-(m-1)},$$

(a) $$y^{n-1} = (-ab^{-1})^{(n-1)/n} \cdot x^{m(n-1)/n},$$

(b) $$x^{m-1} = (-ba^{-1})^{(m-1)/m} \cdot y^{n(m-1)/m}.$$

From (1) and (a) we get

(c) $$r(x,p) \equiv bn^n p^n + am^n x^{m-n},$$

while (1) and (b) give

(d) $$s(y,p) \equiv bn^m p^m + am^m y^{m-n}.$$

Evidently (c) and (d) are the same when $m=n$, and we have $t(p)$.

Example 1.

(1) $$x^2p^2 - 3xyp + 2y^2 + x^3 = 0.$$

Here $h(x, y)\equiv 4x^3-y^2$. Since $m\neq n$, it is not possible to have $t(p)$ as a factor and we must choose between $r(x, p)\equiv p^2-9x$ and $s(y, p)\equiv 2p^3-27y$. Since we would naturally solve (1) for y, $r(x, p)$ must appear as a factor in $\phi(x, p, dp/dx)=0$. Thus,

$$\phi(x,p,dp/dx) = 8x^3[p^2 - 9x][2 - x(dp/dx)].$$

Here

$$\lambda_1 = 8x^3, \ \phi_1(x,p,dp/dx) = 2 - x(dp/dx).$$

Example 2.

(1) $$p^3 - 4xyp + 8y^2 = 0.$$

Here $h(x, y)\equiv 4x^3-27y$. From (c) and (d), we have

$$r(x,p) \equiv 9p - 4x^2, \quad s(y,p) \equiv p^3 - 4y^2.$$

Solving (1) for x, we have

$$\phi(y,p,dp/dy) \equiv [p^3 - 4y^2][p - 2y(dp/dy)],$$

and $s(y, p)$ appears as a factor. Had (1) been solved for y, we should have had

$$\phi(x,p,dp/dx) \equiv p^3[9p - 4x^2][(dp/dx) - 2p],$$

in which $r(x, p)$ appears as a factor, with

$$\lambda_1 = p^3, \ \phi_1(x,p,dp/dx) \equiv (dp/dx) - 2p.$$

Example 3.

$$p^2 - xp + y = 0. \tag{1}$$

Here $h(x, y) \equiv x^2 - 4y$. From (c) and (d), we have

$$r(x,p) \equiv 2p - x, \quad s(y,p) \equiv p^2 - y.$$

Since (1) is linear in both x and y we can solve for either one. Solving for y, we have

$$\phi(x,p,dp/dx) \equiv (2p - x)(dp/dx),$$

and $r(x, p)$ appears as a factor. Solving for x gives

$$\psi(y,p,dp/dy) \equiv (p^2 - y)(dp/dy),$$

and $s(y, p)$ appears as a factor.

Example 4.

$$y^2p^3 + 2xp - y = 0. \tag{1}$$

Here $h(x, y) \equiv 32x^3 + 27y^4$, and (c) and (d) give

$$r(x,p) \equiv 8xp^4 + 3, \quad s(y,p) \equiv 2yp^3 + 1.$$

Solving (1) for x, we shall have

$$\psi(y,p,dp/dy) \equiv 2[2yp^3 + 1][p + y(dp/dy)],$$

and $s(y, p)$ appears as a factor, with $\lambda_2 = 2$. Solving (1) for y leads to

$$\phi(x, p, dp/dx) \equiv 16p^3[8xp^4 + 3][p^6 - 2x(dp/dx)].$$

Here $\lambda_1 = 16p^3$, and $r(x, p)$ appears as a factor.

Example 5.

$$xp^2 - 2yp + ax = 0. \tag{1}$$

Here $h(x, y) \equiv ax^2 - y^2$. From (c) and (d), we have

$$r(x,p) = s(y,p) = t(p) \equiv p^2 - a.$$

Then

$$\phi(x,p,dp/dx) \equiv 2[p^2 - a][p - x(dp/dx)],$$

and $t(p)$ appears as a factor, with $\lambda_1 = 2$; while

$$\psi(y, p, dp/dy) \equiv p^{-1}(p^2 - a)[(p^2 + a) - 2py(dp/dy)],$$

in which also $t(p)$ is a factor, with $\lambda_2 = p^{-1}$.

Case 2.

$$h(x, y) \equiv ax^m + by^n - c = 0.$$

Here

(a) $$p = (-ab^{-1}) \cdot mn^{-1} \cdot x^{m-1}y^{-(n-1)},$$

(b) $$y^{n-1} = [(c - ax^m)b^{-1}]^{(n-1)/n},$$

(c) $$x^{m-1} = [(c - by^n)a^{-1}]^{(m-1)/m}.$$

Using (a) with (b) and (a) with (c), we get

(d) $$r(x, p) \equiv bn^n(c - ax^m)^{n-1}p^n + (-1)^{n-1}(am)^n x^{n(m-1)},$$

(e) $$s(y, p) \equiv (bn)^m p^m + (-1)^{m-1}am^m(c - by^n)^{m-1}y^{m(1-n)},$$

respectively.

It is evident from (d) and (e) that $r(x, p)$ and $s(y, p)$ will never become $t(p)$ unless $m = n = 1$.

Example:

$$(x^2 - 1)p^2 - 2xyp + y^2 - m^2 = 0.$$

Here $h(x, y) \equiv m^2x^2 + y^2 - m^2$. From (d) and (e), we have

$$r(x, p) \equiv (x^2 - 1)p^2 + m^2x^2, \quad s(y, p) \equiv y^2p^2 + m^2y^2 - m^4.$$

Then

$$\phi(x, p, dp/dx) \equiv (p^2 + m^2)^{-1}[(x^2 - 1)p^2 + m^2x^2](dp/dx),$$

and $r(x, p)$ appears as a factor, with

$$\lambda_1 = (p^2 + m^2)^{-1}, \phi_1(x, p, dp/dx) = dp/dx.$$

Also

$$\psi(y, p, dp/dy) \equiv 2p[p^2y^2 + m^2y^2 - m^4](dp/dy),$$

and $s(y, p)$ is a factor with $\lambda_2 = 2p$ and $\psi_1(y, p, dp/dy) = dp/dy$. The fact that p is a constant in each case shows that the equation is of the Clairaut type.

Case 3.

Here $$h(x, y) \equiv ax^m y^n + b = 0.$$

(a) $$p = (-mn^{-1})yx^{-1},$$

(b) $$y = (-ba^{-1})^{1/n} \cdot x^{-m/n},$$

(c) $$x = (-ba^{-1})^{1/m} \cdot y^{-n/m}.$$

Using (a) with (b) and (a) with (c), we get

$$(d) \qquad r(x,p) \equiv an^n p^n + (-1)^n\, bm^n x^{-(m+n)},$$

$$(e) \qquad s(y,p) \equiv bn^m p^m + (-1)^m\, am^m y^{m+n}.$$

Clearly (d) and (e) are not the same unless $m=n=0$: If this should occur $h(x, y)$ would become a constant and $f(x, y, p)$ would be factorable as to p, which was ruled out.

From a study of these three cases we see that $t(p)$ can only occur when $m=n$, and in Case I only are m and n unrestricted in their values.

Example 1.

$$x^3p^2 + x^2yp + a^3 = 0.$$

Here $h(x, y)\equiv xy^2-4a^3$. From (d) and (e) we have

$$r(x,p) \equiv x^3p^2 - a^3, \quad s(y,p) \equiv 8a^3p + y^3.$$

Then

$$\phi(x,p,dp/dx) \equiv x[x^3p^2 - a^3][2p + x(dp/dx)],$$

and $r(x, p)$ is a factor, with $\lambda_1=x$; and $\phi_1(x, p, dp/dx)\equiv 2p+x(dp/dx)$.

Example 2.

$$x^4p^2 - xp - y = 0.$$

Here $h(x, y)\equiv 4x^2y+1$. From (d) and (e), we have

$$r(x,p) \equiv 2x^3p - 1, \quad s(y,p) \equiv p^2 - 16y^3.$$

Then

$$\phi(x,p,dp/dx) \equiv [2x^3p - 1][2p + x(dp/dx)],$$

and $r(x, p)$ is a factor, with $\lambda_1=1$; and $\phi_1(x, p, dp/dx)\equiv 2p+x(dp/dx)$.

COORDINATES USED IN HINDU ASTRONOMY

By SUKUMAR RANJAN DAS, Calcutta, India

Coordinates in astronomy are the quantities by means of which the position in space of a heavenly body is defined. The systems of coordinates used in Hindu astronomy are nearly the same as those used in modern astronomy. Strictly speaking the Indian Astronomical Works do not very clearly point out in a systematic manner how the position of a heavenly body is to be defined with regard to the fundamental planes. The position is indicated whenever occasion arises, but there is no formal statement how the indication is to be made and how the position is to be pointed out. Information on this subject is scattered throughout almost every book, but the major portion is contained

in the chapter on the construction of the armillary sphere, i.e. *Golabandhādhikāra.*

As in modern astronomy, the first system determines the position of a body by its azimuth and altitude or zenith distance. In measuring the azimuth the Indian mode differs from the modern one. In modern astronomy it is the angle between the meridian and the vertical circle through the center of the body under observation, being measured from the north point of the horizon all round through 360° in the direction north, east, south and west. In Hindu astronomy it is the angle which the vertical circle through the centre of the body makes with the prime vertical, being measured either way from the east and west points of the horizon separately. It never exceeds 90° and is north or south from the east or west. Altitude and zenith distance are precisely the same as in modern astronomy.

The only attempt made by Varāhamihira to define the coordinates systematically is in the chapter on the construction of astronomical instruments in *Pañcha Siddhāntikā.* In chapter XIV, he defines the horizon as the circle in which the sky is joined as it were to the earth, in it the east-west and north-south lines are drawn (verse 17). "The interval between the pole and the horizon is the terrestrial latitude; and the difference of the latitude and ninety degrees is called co-latitude, which declines from the zenith towards the pole. The day circle is what intervenes between the rising and setting of the sun etc." (verse 18).

Brahmagupta more explicitly defines the first system of coordinates. He says: "One circle called the *Samamaṇḍala* or the prime vertical has its plane stretching east and west; another stretching north and south is termed the *Yāmyottaravṛtta* or meridian; another called the *Kṣitija* or horizon encircles the other two like a girdle. At the common centre of these circles is the observer located on the earth." (*Brāhma Sphuta Siddhānta, Golādhyāya,* verse 48.)

The azimuth circle is thus described by the same writer: "In as much as an observer on the earth finds a planet in the circumference of the upper half of the azimuth circle (*Dṛgmaṇḍala*) it turns with the planet." (Verse 55, *Golādhyāya.*)

Thus we get the three fixed planes—the meridian, the prime vertical, and the horizon; and these give the first system of coordinates.

One noticeable point is the importance attached to the particular azimuth circles which stand midway between the meridian and the prime vertical bisecting the right angles which they make with each other. Brahmagupta, Lalla and Bhāskara have mentioned this fact. Lalla writes

> Pürvāparamürdhvamadhaḥ prathamaṃ samamaṇḍalaṃ jagur vṛttam/
> Yāmyottaraṃcha tadvattathaiva vṛttadvayaṃ cha vidis'oḥ

Āveṣṭyamānametānyardhachchhedena yadbhavedṛttam/
Taṭ kṣitija maṇḍalaṃ syādudayāstamayāviha dyusadām//2/
(Golabandhādhikāra)

Here the azimuth circles are named as the vertical circles of intermediate directions. The word *vidis'a* literally means "of no direction."

Regarding the use of the first system of coordinates we get the following information in Bhāskara's *Siddhānta S'iromaṇi:* "The arc of the horizon measured in degrees, intercepted between the east point and the point at which the azimuth circle under consideration meets the horizon is called the *digaṃs'a* or azimuth. The *jyā* or linear sine of *digaṃs'a* is termed the *drigjyā*. For the western part of the horizon the measurement is similar." (Gaṇitādhyāya, *Tripras'na*, verse 45.)

About the position of the heavenly body in the azimuth circle or *dṛgmaṇḍala*, i.e. about its zenith distance and altitude, Bhāskara says: "the position of a planet is to be given not by the zenith distance or *nata* and altitude or *unnata* but by *drigjyā* and *s'aṃku*." (*Golabandha, Vāsana bhāṣya*).

The terms *drigjyā* and *s'aṃku* are explained by Brahmagupta who writes, "the linear sine of the zenith distance is the *drigjyā*, that of the altitude is the *s'aṃku*, the altitude and the zenith distance are reckoned in the azimuth circle." (*Brāhma Sphuta Siddhānta, Golādhyāya*, verse 63.)

The relation between altitude and zenith distance is thus given by Bhāskara; "90 − zenith distance = altitude" (Ganitādhyāya, *Tripras'na*, 32.)

The four cardinal points are called the *Uttara-svastika* (north point), *prāk-svastika* (east point), *dakṣiṇa-svastika* (south point) and *apara-svastika* (west point). The zenith and the nadir are the *kha-svastika* and *adhaḥ-svastika*.

The second system of coordinates consists of the hour angle and declination. These are known as the *nataghaṭi* and *sphuta-krānti*, briefly mentioned as *nata* and *krānti*. There is besides, an *unnata-ghati* or time-altitude. It is the angle which the hour circles of a heavenly body at its rise and at a given time make with each other.

Of a star the time-altitude or *unnata-ghati* is the sidereal time elapsed since its appearance on the horizon; for the sun it is measured by clock-time; for the moon or a planet it is the sidereal time elapsed diminished by the change of right ascension during it, neglecting in all cases the change of declination. The *nata-ghati* or hour angle is half day of the body(that is the sidereal time taken by it to pass from the horizon to the meridian) diminished by the time-altitude.

For the western sky the *nata-ghati* is the sidereal time elapsed since the star under observation crossed the meridian and the *unnata-ghati* is the sidereal measure of the time from the moment under consideration to the setting of the star or in other words its half day diminished by the nata-ghati. For other

heavenly bodies change of right ascension is to be taken into account as in the case of eastern sky. The word *ghati* means twenty four minutes. It is either sidereal (*ārkṣa*) or mean solar (*sāvana*). Like the word "hour angle" *nata-ghati* signifies the time equivalent of an angle; *ghati* in this as "hour" in that means time. The following definition of *nata* and *unnata* is given by Bhāskara: "considering the part of the day that is over and the part that yet remains, the less is the *unnata* or time-altitude; half the time diminished by it is called the *nata* or hour angle." (Gaṇitādhyāya, *Tripras'na*, verse 53).

The above verse is intended to indicate the hour angle of the sun, but its application may be extended to include all heavenly bodies. We may for that purpose take day in the sense of the time from the rising to the setting of a heavenly body and reduce it to the standard of a complete revolution round the earth.

The *Mahā-siddhānta* of *Ā*ryabhaṭa II gives a similar definition in the chapter on *Tripras'na* or "Triple Enquiry." He says: "The part of the day that is over is the time-altitude or *unnata* in the eastern sky, while the part that yet remains is the *unnata* in the western sky. Half the length of day diminished by the *unnata* is the *nata* or hour angle."

About declination Bhāskara says that it is the distance of a heavenly body from the equator along its secondary through the body. He says: "The perpendicular distance from the equator of a planet lying in its orbit is its declination."

The equator from which declination is measured lies in relation to the observer as described in Brahmagupta's *Brāhma Sphuta Siddhānta:* "The equator meets the horizon at the cardinal points east and west, the top lying to the south of the prime vertical by the latitude of the observer and the bottom as much to the north of it." (*Golabandha*, verse 51.)

The equator and the meridian are evidently the two fixed planes of reference of this second system of coordinates, though importance is attached to the six hour circle or *unmaṇḍala*. The *unnata* or time-latitude is related to it just as the hour angle or *nata* is related to the meridian, though not reckoned in the same way.

Brahmagupta describes the six-hour circle as below: "The *unmaṇḍala* or six-hour circle meets the horizon at the east and west points and is above and below it from the north and south cardinal points by the latitude of the observer. The increase and decrease in the duration of day and night are due to it." (*Golādhyāya*, 50).

The fixed planes of the two systems are together in Brahmagupta's words: "The prime vertical, the meridian, the horizon, the equator and the six-hour circle which are fixed planes." (*Golādhyāya*, 67).

Of the two modern systems of coordinates which are independent of diurnal

motion, Hindu astronomy frequently makes use of that in which the ecliptic is the primitive circle.

In this system the latitude (*Asphuta-s'ara*, briefly *s'ara*) of a point of the sphere is its distance from the ecliptic (*krānti-vṛitta*) measured on a secondary to it. *Vikṣepaḥ Kadambābhimukho bhavati*, i.e. "latitude is directed towards either pole of the ecliptic," says Bhāskara. The longitude (*sphuta*) is the arc of the ecliptic intercepted between the above secondary and the origin of the coordinates, which is the fixed point marked by the star known as *Revatī* (or Zeta Piscium).

In modern astronomy the origin is the vernal equinox but the Hindu system being sidereal measures longitude from a fixed point, the vernal equinox having its own longitude measured westwards and supposed to increase at a slow rate.

The position of the ecliptic in relation to the equator is thus given in the *Golādhyāya* of Brahmagupta's *Brāhma Sphuta Siddhānta*: "The ecliptic meets the equator at the first points of *Meṣa* (Aries) and *Tulā* (Libra). It is 24 degrees to the north of the equator at the first point of *Karkata* (Crab) and as many degrees to the south at the first point of *Makara* (Goat)." (verse 52).

The obliquity of the ecliptic is therefore, according to Hindu astronomy, 24 degrees.

The last or fourth system of coordinates has the ecliptic for its primitive circle in which the star *Revatī* is the origin. Heavenly bodies are referred to this primitive circle, the ecliptic, by declination circles passing through them. The position is determined by the coordinates called *Dhruvāṃs'a* and *Sphuta-sara*. There are no corresponding terms in modern astronomy. Burgess and Whitney in their translation of the *Sūrya Siddhānta* have named them the "polar longitude" and "polar latitude." The true latitude is called *asphuta-s'ara*.

The last system of coordinates is almost exclusively employed in the observation of stars. When a star is crossing the meridian of an observer its polar longitude and polar latitude can be determined with comparative ease.

It is, however, strange that right ascension in its truest sense is not recognized in Hindu astronomy. No doubt there are rules which enable us to calculate the right ascension of a point in the ecliptic. For example, Āryabhaṭa says: "The linear sine of a given arc of the ecliptic multiplied by the day radius appertaining to the greatest declination and divided by the day radius corresponding to the end of the given arc gives the linear sine of the arc of the equatorial prime vertical from the first point of Aries to the east point of the horizon." This means if l, d, and α be the longitude, declination, and right ascension of a point in the ecliptic, $\sin \alpha = \sin l \cos \omega \div \cos d$, where ω is the obliquity of the ecliptic. (*Golapāda*, aryā 25).

Besides this, portions of the equator are considered according to occasion;

these are called *kālāṃs'as* or degrees of time. But the right ascension of any heavenly body whatsoever or right ascension in the amplitude of its modern application does not appear in Hindu astronomy.

This absence of the recognition of right ascension may be conjectured as an internal evidence of the remote date of the existence of the sidereal system. If the initial point is different from the vernal equinox, simple ascertainment of right ascension is not possible. This might have led to the neglect of such an useful coordinate.

However, it may be said in conclusion that the coordinates used in Hindu astronomy, though not explicitly stated, were more or less the same as are used in modern astronomy.

THE EQUATION OF TIME IN HINDU ASTRONOMY

By SUKUMAR RANJAN DAS, Calcutta, India

The *Sūrya Siddhānta* recognises that the apparent solar day (the interval of time from one sunrise to the next) is variable, as is evident from the following division of time which it gives:

"Time that is measurable is that which is in common use, beginning with the *prāṇa*. The *pala* or *bināḍi* contains six *prāṇas*. The *ghalikā* or *nāḍi* is 60 *palas*, and the *nakṣatra ahórātra*, or sidereal day and night, contains 60 *ghalikās*. A *nakṣatra māsa* or sidereal month, consists of 30 sidereal days."[1]

The same work gives the following rule for finding the length of an apparent solar day in sidereal units: "Multiply the diurnal motion (in minutes) of a planet by the number of *prāṇas* which the sign in which the planet is takes in its rising (at a given place); divide the product by 1800′ (the number of minutes which each sign or *rās'i* of 30° of the ecliptic contains); add the quotient, in *prāṇas*, to the number of *prāṇas* contained in one sidereal day (*i.e.*, $6\times60\times60$, or 21,600 *prāṇas*); the sum will be the apparent day of that planet in *prāṇas*."[2]

Here 1800 is the number of minutes of one *rās'i* or sign of the ecliptic. Let the time taken by a particular sign to rise above the horizon be p *prāṇas*; d, the daily motion of a planet in minutes; and x, the time in *prāṇas* by which a planet's day exceeds the sidereal day, this excess being due to its eastward motion among the stars; therefore, $1800:d::p:x$ or $x=dp/1800$. This is the rule.

This rule is based on the knowledge of two facts: (1) that the daily motion in longitude of any planet is a variable quantity, (2) that the different signs of

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[1] *Sūrya Siddhānta*, chapter V, verses 11 and 12.

[2] *Sūrya Siddhānta*, chapter II, verse 59.

the zodiac take different intervals of time to rise above the horizon at any station, and at any place on the equator.

The *Sūrya Siddhānta* then proceeds to find out the corrections to longitude of planets due to one part of the equation of time: "Multiply the diurnal motion of a planet by the number of minutes contained in the first equation of the sun, and divide the product by the number of minutes contained in a circle, i.e., by 21,600′; add or subtract the quotient, in minutes, according as the sun's equation is additive to or subtractive from the place of the planet, which is found from the *ahargaṇa* at the mean midnight at Lankā; the result will be the place of the planet at the true midnight at Lankā."[1] This is called the *bhujāntara* correction in minutes.

The *bhujāntara* correction is to be applied to the place of a planet found from the *ahargaṇa* for finding the place of the planet at the true midnight at Lankā, arising from that portion of the equation of time which is due to the sun in the ecliptic.

The *Siddhānta S'iromaṇi*[2] of Bhāskarāchārya has, however, laid down more accurate rules:

"Multiply the sun's equation of center by the time the sign of the zodiac, in which the sun is, takes to rise at any place on the equator, and divide the product by 1800; multiply the quotient thus obtained by the daily motion of a planet and divide by the number of *prāṇas* in a whole day; then apply this last quotient positively or negatively to the longitude of a planet, according as the sun's equation of center is applied positively or negatively. This correction is called *bhujāntara*."

These rules will be evident from the way in which the longitude of a planet is calculated. The equation of time is the difference of the right ascensions of the true sun moving along the ecliptic and the mean sun moving along the equator. It has two parts; the difference in the right ascensions of the true sun and a fictitious star moving along the ecliptic with the mean motion of the sun is one part of the equation of time; while the difference of the right ascensions of the mean sun moving along the equator and this fictitious star is the other part. The former is called *bhujāntara* and the latter *udayāntara* in Hindu astronomy. The *Sūrya Siddhānta* is content with only having applied the *bhujāntara* correction. It was Bhāskara who first of all Hindu astronomers detected this *udayāntara* correction.

Bhāskara gives the following proof of *bhujāntara:*

"The longitude that has been found for the mean sunrise is converted to that for the apparent sunrise. First convert the sun's equation of center into

[1] *Sūrya Siddhānta*, chapter II, verse 46.

[2] *Siddhānta S'iromaṇi*, Grahagaṇita, chapter VIII, verse 61.

prāṇas by proportion; if the number of minutes in a sign rises on the equator, in how many *prāṇas* will the arc of the equation of center rise? Then take another proportion; if the number of *prāṇas* of a day changes the longitude of a planet by the amount of daily motion, what is the change for these *prāṇas*? The result in minutes should then be applied negatively or positively as the apparent sun rises before or after the mean sun. Hence the rule is proved."[1]

Then Bhāskara gives the rule for the *udayāntara* correction:

"To the mean sun add the total amount of the precession of the equinoxes; find the sign in which the sun now falls and the degrees of it which are passed over. Multiply the degrees of the sign by that sign when on the equator and divide by 30°. The result is the number of *bhukta-asus* or *prāṇas* taken by that sign. Now add up the *asus* (*prāṇas*) while on the equator of the whole signs passed over by the sun and add to this sum the *bhukta-asus* (or *prāṇas* taken) found above; the result is the number of *prāṇas* of the right ascension of the mean sun. Next reduce the mean longitude of the sun to which the total amount of precession has been added to minutes of arc. Take the difference of these minutes and those *prāṇas*, multiply the daily motion of a planet by it and divide by the total number of *asus* of a whole day; add the result, which is in minutes of arc, to the longitude of the planet if the number of *asus* is greater than the number of minutes and subtract if the minutes are greater in number. This correction is called *udayāntara*."

Bhāskara has given the following proof of his rule:

"The *ahargana* (the number of days elapsed since creation) that has been found elsewhere (for the calculation of longitudes) is of mean solar units (days), as the apparent solar day is variable. The sixty sidereal *ghatis* increased by the number of *asus* (which are equal to the minutes of arc) in the mean daily motion of the sun, is equal to one mean solar day i.e., 60 *ghatis* 59 *prāṇas* and 8 sixtieths (is equal to 24 hrs. 3 min. 56 secs. Twenty-two sixtieths in sidereal time). But the apparent solar day is 60 ghatis and the time in which the arc of the daily motion rises is a variable quantity, because every day the daily motion is different and also because the times of rising of the different signs of different months are different. As the apparent *ahargaṇa* is not obtained in the way described before, we find the mean *ahargaṇa* and we do not get the longitudes at the time of sunrise at Lankā (the City of Rāvana); we get them sometimes for a moment a little before sunrise and sometimes for a moment a little after sunrise. Thus we find the mean *ahargaṇa* when the mean sun is near the horizon at Lankā."

We next proceed to find the difference between the mean and the apparent

[1] See Vasāna Bhāṣya under verses 62, 63, 64 of *Grahaganita*, chapter VIII. See also chapter V, verse 43 and the footnote, *Golādhyāya* (B. D. Sastri's edition).

ahargaṇa. Add up the *asus* at the time of the rise on the equator of the signs passed over by the sun from the beginning of the sign *Meṣa*. The resulting *asus* is the excess of the *ahargaṇa* over the sidereal day. Hence the difference between the *asus* and the minutes of arc is the difference between the two *ahargaṇas*. Now apply the proportion; if the *asus* of a whole day changes the longitude by the daily motion, what will be the change for this difference of *asus*? The result is additive if the *asus* are greater than the minutes, otherwise it is to be subtracted. (See also chapter IV, verses 19, 20, 21, 22, 23 of Golādhyāya and the footnote by Bapu Dev Sastri.)

Bhāskara has given further explanation of his *udayāntara* correction:

"If the ecliptic be divided into four quadrants beginning from the vernal equinox, each quadrant will rise on equator in 15 *ghatis* (or 6 sidereal hours); but each sign will not rise in 5 *ghalis* (2 sidereal hours) and the *udayāntara* correction goes on increasing till the middle point of the first quadrant, and then continues to decrease. Hence at the end of each quadrant it vanishes (i.e., four times a year) and attains the greatest value at the middle point of each quadrant. The above is a rough method and meant for a beginner. The method by which the *udayāntara* correction is more accurately obtained is as follows: to the mean longitude of the sun add the total amount of the precession of the equinoxes, now find the *jyā* of the longitude ($R \times$sine longitude) and the *kotijyā* of the corresponding declination ($R \times$cosine declination); multiply this *jyā* by the *kotijyā* of the declination of the last point of the third sign and divide by the *kotijyā* of the declination. Find the arc in *asus* of which this is the *jyā;* and diminish, by these *asus*, the number of minutes in the mean longitude of the sun as increased by the total amount of precession; the result is the correct number of asus of *udayāntara*. In this way at the middle of the quadrant the *udayāntara* becomes a little greater than 26 *palas*" (i.e., 10 minutes 24 seconds).

Now if this *udayāntara* correction is so necessary, being that part of the equation of time due to the obliquity of the ecliptic, why was it ignored by the former astronomers? Bhāskara replies by saying that, "after all, this correction is a variable and a small quantity and vanishes at the four ends of the four quarter years." Kamalākarabhatta in his Siddhāntatattvaviveka tried to ignore this *udayāntara* correction, but his arguments are not very convincing.[1] Bhāskara is the only Indian astronomer who found out the two parts of the equation of time.

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[1] See page 251 of *Bhāratīya Jyotiḥ Śāstra* by S. B. Dikshit.

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. STROPHOIDAL CURVES AND CUBICS

By R. M. MATHEWS, West Virginia University and OTTO DUNKEL, Washington University

1. Problem 2928 [1921, 466] states the following theorem: *A line drawn through the end, F, of the loop of a right strophoid or of a folium of Descartes cuts the curve again in Q and R such that the segment QR subtends a right angle at the node. This article will furnish answers to the questions raised in that problem, some references to the history of the curves possessing this property, and the development of certain other properties.*

This property of the right strophoid can be traced as far back as Barrow's *Lectiones Geometricae* (1669), p. 69; and for the oblique strophoid to Quetelet and Dandelin.[1] For these curves the locus of the centers of the circles on QR as diameter is a straight line.

Barbarin[2] has generalized the notion by defining strophoidal curves as follows: we are given two fixed points O and F and a curve Γ; for each point K on Γ we take Q and R on KF such that KQ and KR are each equal to KO; the locus of Q and R is a strophoidal curve of Γ.

2. The Equation of Strophoidal Curves. If F be the pole, FO the polar axis with $FO=a$, and Γ be defined by $\rho_1=f(\theta)$, then the equation of the locus is

$$\rho^2 - 2\rho f(\theta) - a^2 + 2a\cos\theta f(\theta) = 0.$$

Barbarin found some general properties of such curves and later De Longchamps[3] discussed the construction of their tangents.

For cartesian coördinates take O as origin with F on the x axis at $(a, 0)$. Let the locus of K be $f(\alpha, \beta)=0$. Then the equations of the line KF and of the required circle are respectively

$$\alpha y + \beta(a - x) - ay = 0,$$
$$2\alpha x + 2\beta y - (x^2 + y^2) = 0.$$

From these we find

$$\alpha:\beta:1 = [(x - a)(x^2 + y^2) + 2ay^2]:y(x^2 + y^2 - 2ax):2(x^2 + y^2 - ax),$$

[1] *Mémoire sur quelques propriétés de la focale parabolique*, Nouveaux Mémoires de l'Académie Royale des Science et Belles-lettres, Bruxelles vol. 2, (1882), pp. 169–202.

[2] *Courbes strophoïdales*, Revue de Mathématiques Spèciales, vol. 2, (1892–94), pp. 298–299.

[3] *Sur les strophoïdales*, Mathesis (2), vol. 4, (1894), pp. 138–141.

and on inserting these values of α and β in the equation $f(\alpha, \beta) = 0$ of Γ, we have the equation of the locus desired.

When Γ is a straight line the locus is a circular cubic; that is, it passes through the circular points at infinity. It is obvious from the equations that it passes through O and F. Moreover, it follows that F is a focus of the cubic. The foci of a curve are defined as the intersections of the tangents to the curve which pass through the circular points at infinity. If I and J denote the circular points at infinity, then on the line FI we have R at I. The perpendicular to OI is OI itself, and hence Q coincides with R. Hence FI and FJ are tangents to the cubic at I and J, and we may for this reason call F a singular focus. Thus strophoidal circular cubics must contain their singular foci. These curves have been studied synthetically by Lagrange.[1]

If the line Γ passes through a point common to the circles of the pencil, the curve is an oblique, or general strophoid. The singular focus F is at the end of the loop only for the right strophoid or Booth's logocyclica. Take Γ as the y-axis, $\alpha = 0$, and a as negative, and the usual equation for this curve results.

If the locus of centers Γ be other than a straight line, the degree of the strophoidal curve will be higher than three, in general. It may be three for special cases as is indicated by the fact that the property holds for the folium which is not a circular cubic. We shall now find the form of the equation for the general cubic.

3. Strophoidal Cubics. Let a fixed straight line cut a cubic in the points O, S, F, and let any other line through F cut the curve in the variable points Q and R We shall determine the form of the equation so that $\angle QOR$ shall be a right angle.

With axes as before, S at $(b, 0)$, finite, and F at $(a, 0)$, the equation of the cubic may be written

$$Ax(x - a)(x - b) + Bxy^2 + Cyx^2 + Dy^3 + A'xy + B'y^2 + C'y = 0,$$

where neither A nor a is zero. Writing the equation of FQR in the form $\lambda y = x - a$, we eliminate a factor $(x - a)$ between the two equations and obtain

$$\lambda Ax(x - b) + Bxy + Cx^2 + Dy^2 + A'x + B'y + C' = 0.$$

On making this quadratic equation homogeneous by use of the factor $1 = (x - \lambda y)/a$, we have as the equations of the lines OQ and OR

$$(a^2D - \lambda aB' + \lambda^2C')y^2 + (aA' - \lambda abA + \lambda a^2A + a^2C + C')x^2 + (\lambda^2abA + a^2B - aA'\lambda + aB' - 2C'\lambda)xy = 0.$$

Since the product of the slopes of these two lines is to be -1, the sum of the coefficients of x^2 and y^2 must be zero; thus

[1] *Sur les cubiques strophoïdales*, Nouvelles Annales de Mathématiques, (3), vol. 1 (1900), pp. 66–74.

$$C'\lambda^2 + a(aA - bA - B')\lambda + (aA' + a^2C + C' + a^2D) = 0.$$

Since this is to be true for all values of λ, we must have

$$C' = 0, \quad B' = (a - b)A, \quad A' = -a(C + D),$$

and the equation of the cubic is

$$(1) \quad A[x(x - a)(x - b) + (a - b)y^2] + Bxy^2 + Cxy(x - a) + Dy(y^2 - ax) = 0.$$

When S is at infinity on OF, we write the equation as before but in it we replace $(x-b)$ by unity. We obtain in the same way for this case

$$(2) \qquad A[x(x - a) + y^2] + Bxy^2 + Cxy(x - a) + Dy(y^2 - ax) = 0.$$

The constants A, B, C, D, a, b are arbitrary subject to the conditions $A \neq 0$, $a \neq 0$.

If O is a double point the two branches of the curve through it must cross at right angles, as we see from the defining property of the curve. Suppose now that F and O are ordinary points for the curve. Then when R falls on S, Q must fall on O since it must lie on the intersection of FOS and the perpendicular to this line at O. Thus the perpendicular OT to OF at O is the tangent at O. Suppose the tangent at F cuts this perpendicular in T; then T must be a point on the curve which corresponds to F. These facts are also easily seen from the equations. For, from equation (1) we see at once that the y-axis is tangent to the curve at the origin and cuts it again at $T[0, A(b-a)/D]$. Again, we find that the slope of the tangent at F is $A(a-b)/aD$, which is the slope of the line TF. Thus O and F are "conjugate points" with their tangents meeting on the curve in T. Accordingly, if two ordinary points are to be a basis for the strophoidal property on a cubic it is necessary that they be conjugate points and that their chord be normal to the tangent at one of them.

4. The Locus of Centers for a Strophoidal Cubic. We proceed now to find the locus $f(\alpha, \beta) = 0$ of the mid-point K of QR. In the first case we eliminate y between $\lambda y = x - a$ and the cubic (1). Then the abscissas x_1 and x_2 of Q and R are the roots of the quadratic

$$(A\lambda^3 + C\lambda^2 + B\lambda + D)x^2 + [-bA\lambda^3 - a(C + D)\lambda^2 + A(a - b)\lambda - aB\lambda - 2aD]x + [a^2D - aA(a - b)\lambda] = 0.$$

Since $2\alpha = x_1 + x_2$, we have

$$2\alpha(A\lambda^3 + C\lambda^2 + B\lambda + D) + [-bA\lambda^3 - a(C + D)\lambda^2 + A(a - b)\lambda - aB\lambda - 2aD] = 0.$$

On eliminating λ between this equation and $\lambda\beta = \alpha - a$, we get for $f(\alpha, \beta) = 0$

$$(3) \quad A(b - 2\alpha)(\alpha - a)^2 + [a(C + D) - 2C\alpha](\alpha - a)\beta + (aB - aA + bA - 2B\alpha)\beta^2 - 2D\beta^3 = 0.$$

In the second case we find

$$(4)\quad A(\alpha - a)^2 + [2C\alpha - aC - aD](\alpha - a)\beta + (2B\alpha + A - aB)\beta^2 + 2D\beta^3 = 0.$$

These equations show that a cubic arises as a strophoidal curve only from a directrix Γ curve which is a cubic with a singular point at the proposed focus F. Indeed these equations include the circular cubics, for with $B=A$ and $C=D$ we get for Γ

$$[A(b - 2\alpha) - 2C\beta][(\alpha - a)^2 + \beta^2] = 0,$$

or a straight line and the null circle at F.

5. In closing we note certain conditions for the degeneration of the strophoidal curve. If n be the order of the curve Γ of centers K, the order of the strophoidal curve S will be $3n$ in general. For a straight line from F will cut Γ in n points K each of which will give two points Q and R of S on this straight line. The multiplicity of F will be n since the perpendicular bisector of FO will cut Γ in n points K such that $KO=KF$. Hence a straight line through F different from these KF lines cuts S in $2n+n=3n$ points.

Suppose now that a pair of these points on the perpendicular bisector happens to be conjugate points K', K'' on the minimal lines through F. Then the circle with K' as center passing through O is satisfied by every point on FK'. The same reasoning applies to FK''. Now the two lines FK' and FK'' are defined by the equation of the null circle at F. Hence a factor must separate from the equation for S which set equal to zero gives this null circle. If the multiplicity of K' and K'' is i, then the separation of the corresponding factors reduces the degree of S by $2i$.

Again, when Γ passes through F, then for the circle of center F and radius FO, every pair of diametrically opposite points Q and R are collinear with F and subtend a right angle at O. Therefore, if F be of multiplicity r on Γ, a factor must separate which equated to zero gives r of these circles, and the degree of S is reduced by $2r$. Thus by the separation of these two kinds of factors, the degree of S is reduced to $3n-2(i+r)$.

It is easily seen from the equations (3) and (4) that $i=1$, and, as we have already observed, $r=2$. Hence the degree of S in this case is $9-2(3)=3$. Moreover, these conditions imposed upon a cubic equation enable us to arrive at the equations (3) and (4).

II. Logarithms of Large Numbers[1]

By C. C. Camp, University of Nebraska

1. Briggs' tables to fourteen places are extremely scarce and the Tables du Cadastre and those of Sang are still unpublished. The scarcity of Vlacq's and

[1] Read before the Illinois Section of the Mathematical Association of America, May 13, 1927.

Vega's tables has been compensated for by the new ten place tables of Peters, but the demand for larger tables still exists. So great is this demand that the French and Germans are preparing tables to 14 and 15 decimals, respectively, while the British, not to be outdone, are preparing a standard table to 20 figures.[1] When a table is available there remain two possible difficulties. One is the obvious inconvenience due to the size of the book and the other is the excessive amount of numerical work still necessary in computing the logarithm of a number with ten or more significant digits. It is with a view to overcoming these that the author became interested in comparing the various methods which have been employed in finding the logarithms of large numbers.

2. A simple method used by Briggs involves a very small so-called brief table which may be illustrated as follows. Separate π into the factors[2]

$$3,\ 1.04,\ 1.0^{2}6,\ 1.0^{3}9,\ 1.0^{4}1 \text{ and } 1.0^{5}52171,$$

where $1.0^{s}m$ implies s zeros between the decimal point and the number m. If one desires a ten place logarithm, the logarithms of the factors may be taken from the table to twelve decimals. For the last factor, if the natural logarithm is sought one simply takes the decimal part. For the common logarithm one multiplies the decimal part of this factor by the modulus M using a multiplication table, usually given in an auxiliary table. Such a brief table is found in Schroen's *Logarithmen* on the last page. It extends to 16 decimals. A similar table is given by Hedrick in the *Macmillan Tables*, to 15 places.[3]

3. Fédor Thoman modifies this process so as to obtain a logarithm correct to 26 figures "by an easy and simple process without division, without interpolation and without a formula"! He points out that it is no fault of the ordinary seven place table that the result of dividing 321473×819255 by 452604×595118 gives 0.9777831 instead of 0.9777822. In such calculations it is safer to use an extra digit or two in the logarithms in order to be sure of the answer. His method will be illustrated by a 15-place logarithm altho his tables run to 27 decimals. To get $\log \pi$, he uses the multipliers[4]

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[1] *Logarithmetica Britannica*, by A. J. Thompson, Parts IX and VIII have now appeared, containing numbers from 80,000 to 100,000 and some auxilliary tables.

[2] To separate a number into such factors, divide the number by the first significant digit unless it is unity, in which case divide by the number represented by the first two significant digits regardless of the number of zeros between; continue this process, separating successive quotients until there are half as many zeros after unity (indicated by s) as the number of figures in the mantissa of the desired logarithm. The divisions may be done by a machine, a multiplication table, or by simple short division.

[3] See a table by D. H. Lehmer, this Monthly, vol. 32 (1925), where such factors are used to compute cube and fifth roots.

[4] The purpose of multiplying by 32 is to obtain a number beginning with 10. The table gives reciprocals of all two-digit numbers, also their logarithms. The other factors are used to create more zeros and so to increase s. Logarithms of reciprocals of such factors are included to 27 decimals.

$$.32,\ 1-.0^{2}5,\ 1-0^{3}2,\ 1-.0^{4}8,\ 1-.0^{5}3,\ 1-.0^{7}3,$$

obtaining $1.0^{8}76,30152,82$. The logarithm of the last number is found from this table, two digits of the decimal part being taken at a time. He then adds the logarithms of the reciprocals of the multipliers taken out to 17 decimals. The whole calculation appears on one page with the result, 0.49714, 98726, 94134.

4. An alternate method is due to Flower who used multipliers of the form $1.0^{*}m$. For example to find $\log \pi$, divide by 4 and multiply successively by

$$1.2,\ 1.05,\ 1.01,\ 1.0^{3}5,\ 1.0^{5}2,\ 1.0^{6}5,\ 1.0^{7}4,\ 1.0^{8}8,\ 1.0^{9}4,\ 1.0^{10}1 \text{ and } 1.0^{11}4.$$

The multipliers[1] are so chosen that the successive products approach unity from below. The final product in this case is $1-x$, where x is $.0^{12}4105140237$. If S denotes the sum of the logarithms of the eleven multipliers, then $\log \pi = \log 4 + \log (1-x) - S$.

5. The method of Briggs has been extended by Gray to 24 decimal places and to factors which contain three significant figures besides unity. His book is scarce.

6. Steinhauser carries the same method to factors containing blocks of four digits. To illustrate,

$$\pi = 3.141 \times 1.0^{3}1886 \times 1.0^{7}8307 \times 1.0^{11}5638,\ 2819,\ 6-.$$

The last number may be separated into the factors $1.0^{11}5638$, $1.0^{15}2819$ and $1.0^{19}6-$, which are sufficiently accurate for a twenty place logarithm. The first three factors are found in tables A, B, C, respectively, the others in table D, which is really a multiplication table for the modulus. Peters' four by two multiplication table or a calculating machine would expedite the above divisions. An extension of Flower's method would be a distinct advantage here. To determine the first multiplier one may use the reciprocal of the first four digits from Barlow's table, taken to four digits and diminished if necessary.

One may guard against errors by factoring the given number in two independent ways. Some such check is valuable no matter what tables are used.

If one desires a logarithm to seventeen decimals, two divisions or multiplications are sufficient. In such a case the first three factors are found in tables A, B, C and the fourth in D.

7. The method of factoring is combined with simple interpolation for the second factor in Scott's table. Incidentally an error was found in log 1.00038

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[1] The rule is to divide by the first digit plus one, then to multiply by a factor beginning with unity and having another significant digit obtained by subtracting the multiplicand, digit by digit, from 9 until a remainder is not zero. One might have used 1.06 instead of the two factors 1.05 and 1.01. Then the next multipliers would have been $1.0^{2}9$, $1.0^{4}7$, etc. If natural logarithms are sought this method requires the briefest table.

in his illustrative problem: to find log 8.463,298,746. The factors are 8.46 and 1.0003899227. The result given is 0.9275396714. Andoyer in his thirteen-place table carries the same method to second differences. Crelle's multiplication table is used and the second factor is taken to 14 places. The first and second differences in this case are 43412734 and 433, respectively.

8. One may use a corrected first difference instead of a second difference with Peter's table but by either method it may be surprising that it requires 4 or 5 minutes to take out the logarithm of a ten-digit number. The last decimal is always in doubt. A seven place logarithm table or a calculating machine is a valuable aid in the interpolation.

9. Some of the methods mentioned above may be applied in Thompson's table. Eventually it will have the advantage for five-figure numbers when the logarithm is required to more than ten places and up to 20. Extra work is now necessary to bring numbers within the range of the table. If one divides by the first five digits it is possible to interpolate in an auxiliary table for the second factor. First differences are sufficient for a 14-decimal logarithm, while second differences are needed to obtain a logarithm to 20 places. A preferable method for the latter is first to calculate by interpolation, using central differences, the logarithm of the first eight digits, then to divide the so called transformed number by the latter. Either method requires extensive interpolations with Everett's "coefficients" and withal a tremendous amount of arithmetic.

10. The most complicated method found in any brief table was that of Börgen, involving antilogarithms to 10 or 11 places corresponding to four-decimal logarithms. His table is based on something analogous to addition logarithms. Let us illustrate by taking the number $A = 8463298746$, previously treated. One finds in the first table the antilogarithm $B = 8462525688.1$ such that $A > B$. Here $\log B = .9275$, (one disregards characteristics). Find a second number $B_1 < A - B = 773{,}057.9$ such that its logarithm has the same fourth digit 5, namely $B_1 = 771{,}791.5$. Log $B_1 = .8875$. Perform the process once more finding $B_2 = 1{,}266.2$ such that $\log B_2 = .1025$. The residue $A - (B + B_1)$ is 1,266.4. Hence there is a remainder of .2. If we consider

$$A = B + B_1 + B_2 + B_3 = B(1 + B_1B^{-1}) \cdot (1 + B_2B^{-1}) \cdot (1 + B_3B^{-1}) \cdot (1 - B_1B_2B^{-2}),$$

approximately, then we have A factored. Write $L_1 = \log B - \log B_1 = .040$ and $\Delta L_1 = \log (1 + B_1B^{-1})$, and similarly for L_2, L_3. Then $\log A = \log B + \Delta L_1 + \Delta L_2 + \Delta L_3 - \Delta(L_1 + L_2)$, approximately. The last two items are found by the approximation formulas $B_3\Delta L_B$, and $-\Delta L_1 \Delta L_2 / M$ with corrected decimal point. There are two cases for determining the position of the decimal point in the items taken from table B: Case I, $\log B < .637$; Case II, $\log B > .637$; and the items are taken as follows:

$\log B =$	.927 5000 000		from Table A
$\Delta L_1 =$	396 063.2		from Table B, column 1,
$\Delta L_2 =$	649.8		from Table B, column 2,
$B_3 \Delta L_B =$	0.1	$= B_3(.514)$	from Table B, column 2,
$-\Delta(L_1+L_2) =$	−0.1		
	.927 5396 713.2		

The result is quite accurate, but the method seems very awkward and difficult to master.

A comprehensive bibliography of logarithmic tables has recently appeared as No. XIII of *Tracts for Computers*, published by the Cambridge University Press. By a comparison of all the methods one is led to believe that the simplest method for computing a ten place logarithm is that of Steinhauser[1] in connection with Schroen's *Tafel* I. A more extensive logarithm can easily be calculated with a slight increase in the arithmetic. Below is the total work necessary to find log 8.463298746 when one writes down even the work of interpolation in the seven place table:

8463,298746	1.000,0353,003
.4752934	
87	.9275 2434 0875
.4753021	
.9275243	1533 0325
9.5477778	
47	0130
31	.9275 3967 13

III. On Definite Algebraic Quadratic Forms

By Leonard M. Blumenthal, Johns Hopkins University

1. Introduction. A fundamental theorem in the theory of algebraic quadratic forms is that in which necessary and sufficient conditions that the form be definite are obtained.[2] It is the purpose of this paper to obtain conditions for the case of properly definite, non-singular forms by a method involving only elementary considerations. The usual proofs (which this paper avoids), though simplified somewhat for the special case treated here, are believed to be not so

[1] Steinhauser, Logarithmen-Hilfstafeln, Wien, Druck und Verlag von Carl Gerold's Sohn, is still listed for sale by G. E. Stechert and Co., New York, N. Y.

[2] See L. E. Dickson's *Modern Algebraic Theories*, Chapter 4, and Maxime Bôcher's *Introduction to Higher Algebra*, p. 150; see also Ex. 4, 5, p. 147.

simple and straightforward as the inductive method that this paper employs.

2. We seek to establish the following

THEOREM: *A necessary and sufficient condition that a real, non-singular, algebraic quadratic form of matrix* $(a_{ij})(i, j=0, 1, 2, \cdots, n-1)$ *be properly definite is that the signs of*

$$\delta_1 = a_{00},\ \delta_2 = \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix}, \cdots, \delta_n = |a_{ij}| \quad (i,j = 0,1, \cdots, n-1)$$

be all positive or alternately negative and positive according as the form is positively or negatively definite.

We first consider the ternary form

$$(1) \qquad y = \sum_0^2 a_{ij}x_ix_j = a_{00}x_0^2 + 2(a_{01}x_1 + a_{02}x_2)x_0 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2,$$

a quadratic in x_0. A necessary and sufficient condition for (1) to be properly definite is that for all real values of $(x_1, x_2) \neq (0, 0)$ we have

$$(a) \qquad a_{00}(a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2) - (a_{01}x_1 + a_{02}x_2)^2 = \rho^2,$$

$$(b) \qquad a_{00} = \rho,$$

where ρ represents real numbers generically and is positive or negative according as we consider positive or negative definite forms.[1] Writing (a) as a quadratic in x_1 we obtain

$$(c) \qquad D_{11}x_1^2 + 2D_{12}x_1x_2 + D_{22}x_2^2 = \rho^2,$$

where

$$D_{ij} = a_{00}a_{ij} - a_{i0}a_{0j} \quad (i,j = 1,2).$$

In order that (c) be satisfied, it is necessary and sufficient that

$$D_{11} = \rho^2;\ \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \rho^4;$$

i.e., the leading coefficient and the discriminant must be positive. But

$$D_{11} = \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix} = \delta_2,$$

the first principal minor of order two in the determinant of the form, and[2]

[1] If $a_{00} \neq 0$ and $(x_1, x_2) \neq (0, 0)$, the form (1) will vanish for some real x_0 unless condition (a) holds. If $a_{00} = 0$, there will be a real x_0 for which form (1) vanishes unless the coefficient $a_{01}x_1 + a_{02}x_2 = 0$ for every x_1, x_2. But this coefficient vanishes identically only if $a_{01} = a_{02} = 0$. This case is excluded since the form is, by hypothesis, non-singular.

The equations involving ρ are *sign* equations. In a way, the letter ρ does not represent a magnitude but a sign.

[2] Kowalewski, *Einführung in die Determinanten-Theorie* (1909), p. 83.

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \begin{vmatrix} a_{00}a_{11} - a_{10}a_{01}, & a_{00}a_{12} - a_{10}a_{02} \\ a_{00}a_{21} - a_{20}a_{01}, & a_{00}a_{22} - a_{20}a_{02} \end{vmatrix}$$

$$= a_{00}\begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}.$$

Thus conditions (a), (b) become

$$a_{00} = \delta_1 = \rho\ ;\ D_{11} = \delta_2 = \rho^2\ ;\ \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \rho^4\ ;$$

whence $\delta_1 = \rho$; $\delta_2 = \rho^2$; $\delta_3 = \rho^3$.

To procede to the general case, we assume the conditions,

$$a_{00} = \delta_1 = \rho\ ;\ \delta_2 = \rho^2\ ;\ \delta_3 = \rho^3\ ;\ \cdots\ ;\ \delta_n = \rho^n,$$

to be necessary and sufficient for the definiteness of a non-singular quadratic form in n variables and we consider the $(n+1)$-ary form

$$y = \sum_0^n a_{ij}x_ix_j.$$

In order that

$$a_{00}x_0^2 + 2(a_{01}x_1 + \cdots + a_{0n}x_n)x_0 + a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + a_{nn}x_n^2 = \rho,$$

it is necessary and sufficient that for all real $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$,

$$(a') \quad a_{00}(a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + a_{nn}x_n^2) - (a_{01}x_1 + \cdots + a_{0n}x_n)^2 = \rho^2,$$

$$(b') \quad a_{00} = \rho.$$

We write (a') in the form,

$$(c') \quad D_{11}x_1^2 + 2D_{12}x_1x_2 + \cdots + D_{nn}x_n^2 = \rho^2,$$

with the D_{ij} defined as before for $(i, j = 1, 2, \cdots, n)$.

Now (c') is a form in n variables $(x_1, x_2, \cdots, x_n)$. Hence, by hypothesis, the condition that it be definite (positive) is

$$K_1 = \rho^2\ ;\quad K_2 = \rho^4\ ;\ \cdots\ ;\ K_n = \rho^{2n}$$

where $K_i (i = 1, 2, \cdots, n)$ are the principal minors of its determinant. But[1]

$$K_1 = D_{11} = \delta_2\ ;\quad K_2 = a_{00}\delta_3\ ;\ \cdots\ ;\ K_n = a_{00}^{n-1}\delta_{n+1}\ ;$$

whence conditions (a'), (b') may be written

$$\delta_1 = a_{00} = \rho\ ;\quad \delta_2 = \rho^2, \delta_3 = \rho^3,\ \cdots\ ;\ \delta_{n+1} = \rho^{n+1}\ ;$$

and the theorem is proved.

[1] Kowalewski, loc. cit., p. 83.

It may be observed that since the order of the variables is, of course, immaterial, a necessary and sufficient condition that a form be properly definite is that (1) every principal minor determinant be positive for positively definite forms, (2) that every principal minor determinant be plus or minus according as its order is even or odd, for negatively definite forms.

IV. A Reading Course in Mathematics

By Tomlinson Fort, Lehigh University

The Mathematics Department at Lehigh University is making an experiment[1] which we regard with much interest. We have established a course which is described in the catalogue as follows:

"Math. 15. Reading Course in Mathematics. Credit not to exceed one hour per semester, total credit not to exceed three hours, approval of program and written report required."

Permission to give such a course was obtained from the faculty in 1927 and it was advertised on the Mathematics bulletin board. At the beginning of the second semester of 1927–28 it was elected by one senior, three sophomores, and eight freshmen. The distribution is interesting. Even this amount of interest in a course that was not yet in the catalogue was gratifying, particularly as it was in each instance in addition to an otherwise full program, and was at Lehigh where the majority of the students are studying engineering. No student who had made less than a "B" in mathematics the previous semester was eligible. The senior was assigned to Professor Lamson, the freshmen to Professor Smail, and the sophomores to me. We undertake to advise each student separately as to mathematical reading suitable in amount and difficulty to his advancement and ability. The student undertakes to report to his adviser once in two weeks, more frequently if he desires, and to submit a brief written report each time he completes an assignment. These assignments are made after the freest possible discussion with him. The work is going very well. The enthusiasm of the students varies, but on the whole there is more enthusiasm than we expected.

Each adviser is provided with a list of books which are in our library and which seem, in part at least, readable by an undergraduate and likely to interest him. He has this before him when talking the matter over with the student. And inasmuch as teachers at other institutions may be interested in establishing such a course themselves, this list is given here. Of course, it is not comprehensive. We shall add to it and probably strike off some that are on it. It is interesting to know, however, that in a library of moderate size so much could

[1] This experiment at Lehigh University, which involves collateral reading on the part of the student, was begun and the list of readings was prepared before the "Committee on Collateral Reading" of the Mathematical Association of America published its report in the May, 1928 issue of this Monthly.

be found which can be read by an undergraduate. We feel free to recommend for reading individual chapters or even pages. The chief thing to watch is that the student does not get too far beyond his depth. The list is:

Memorabilia Mathematica, by R. E. Moritz; *The Nature of Mathematics*, by P. E. B. Jourdain; *An Introduction to Mathematics*, by A. N. Whitehead; *The Story of Mathematics*, by D. Larrett; *Thinking About Thinking*, by C. J. Keyser; *The Human Worth of Rigorous Thinking*, by C. J. Keyser; *Was ist Mathematik*, by L. Heffter; *The Organization of Thought*, by A. N. Whitehead; *Introduction to Mathematical Philosophy*, by Bertrand Russell; *Lectures on the Philosophy of Mathematics*, by J. B. Shaw; *Philosophy of Mathematics*, by C. J. Keyser; *Science and Hypothesis*, by H. Poincaré; *Science and the Modern World*, by A. N. Whitehead; *The Value of Science*, by H. Poincaré; *Fundamental Concepts of Algebra and Geometry*, by J. W. Young; *Foundations of Geometry*, by D. Hilbert; *Euclid's Parallel Postulate*, by J. W. Withers; *The Thirteen Books of Euclid's Elements*, by T. L. Heath; *Euclid and his Modern Rivals*, by C. L. Dodgson; *Ruler and Compasses*, by H. P. Hudson; *Non-Euclidean Geometry* by H. S. Carslaw; *Non-Euclidean Geometry*, by H. P. Manning; *Non-Euclidean Geometry*, by D. M. Y. Sommerville; *Theory of Parallels*, by N. I. Lobachevski; *Four Dimensional Geometry*, by H. P. Manning; *A Short Account of the History of Mathematics*, by W. R. W. Ball; *History of Mathematics*, by Florian Cajori; *History of Mathematics*, by D. E. Smith; *Greek Mathematics*, by T. L. Heath; *The Development of the Sciences*, by the Yale Faculty; *Makers of Science*, by I. B. Hart; *History of Science*, by Walter Libby; *Ten British Mathematicians*, by Alexander MacFarlane; *History of Mathematics*, by E. W. Brown (Scientific Monthly, May 1921); *History of Mathematics in Europe*, by J. W. N. Sullivan; *History of Statistics*, by John Koren; *Mathematics*, by G. A. Bliss (Scientific Monthly, April 1927); *Too Little Mathematics and Too Much*, by E. B. Wilson (Science, Jan. 20, 1928); *Henry Briggs and His Work on Logarithms*, by A. J. Thompson, (This Monthly, Vol. 32), *Relativity, Special and General Theory*, by A. Einstein; *Relativity*, by C. P. Steinmetz; *The Origin, Nature and Influence of Relativity*, by G. D. Birkhoff; *The Reign of Relativity*, by R. B. Haldane; *Einstein, the Searcher*, by A. Moszkowski; *Mathematical Reality*, by J. B. Shaw; *College Algebra*, by H. B. Fine; *A Survey Course in Mathematics*, by N. J. Lennes; *Engineering Mathematics*, by C. P. Steinmetz; *Lectures on Mathematics*, by F. Klein; *Monographs on Modern Mathematics*, by J. W. A. Young; *Calculus of Variations*, by G. A. Bliss; *Statistics*, by H. L. Rietz; *Analytic Functions of a Complex Variable*, by D. R. Curtis; *Mathematicians and Music*, by R. C. Archibald (This Monthly, Vol. 31); *Bio-mathematics*, by W. M. Feldman; *Mathematical Methods in Economic Research*, by C. C. Morris (This Monthly, Vol. 31); *The Mathematical Theory of Economics*, by G. C. Evans (This Monthly

Vol. 32); *Mathematical Preparation for Physical Chemistry*, by F. Daniels; *Differential Equations in Applied Chemistry*, by F. L. Hitchcock and C. S. Robinson; *The Mathematics of Biology*, by O. Richards (This Monthly, Vol. 32); *Vector Analysis*, by J. B. Coffin; *Vector Analysis*, by Willard Gibbs, E. B. Wilson; *Elementary Vector Analysis*, by C. E. Weatherburn; *Introduction to Quaternions*, by P. Kelland and P. G. Tait; *Elements of Quaternions*, by G. H. Hardy; *Empirical Formulas*, by T. R. Running; *Synthetic Projective Geometry*, by D. N. Lehmer; *Mathematical Theory of Limits*, by J. B. Leatham; *Introductory Mathematical Analysis*, by T. Walmsley; *Life Assurance Primer*, by H. Moir; *Curve Tracing*, by W. W. Johnson; *Theory of Numbers*, by R. D. Carmichael; *Graphical Methods*, by T. R. Running; *HigherMathematics*, by Mansfield Merriman and R. S. Woodward.

V. Two Mutual Relations Between Pairs of Tetrahedra

By Paul Wernicke, Washington, D. C.

There may exist between two tetrahedra a certain reciprocal relation, which we call *orthogrammacy* since the shorter words orthography and orthographic have been preëmpted to designate a type of projection.

The four vectors a, b, c, d, representing by their lengths the areas of the sides of a tetrahedron $ABCD$, to which they are outward normals, will, if placed end to end, form a closed figure, a tetragram in space. For, taking D as origin and denoting the vectors DA, DB, DC by α, β, γ, the sides will be represented by the vector products (in Gibb's notation):

$$2a = \beta \times \gamma,\ 2b = \gamma \times \alpha,\ 2c = \alpha \times \beta,$$
$$2d = (\gamma - \alpha) \times (\beta - \alpha) = -(\beta \times \gamma + \gamma \times \alpha + \alpha \times \beta),$$

so that $a+b+c+d=0$. Their tetragram *abcd* defines by its vertices and by the planes laid through its four pairs of consecutive sides, a second tetrahedron the sides of which are represented by the vectors: $4a \times b = \gamma v$, $-4b \times c = \alpha v$, $4c \times d = (\alpha - \beta)v$, $-4d \times a = (\beta - \gamma)v$, where v is the (pseudo-) scalar product $[\alpha\beta\gamma]$ which is the six-fold volume of tetrahedron $ABCD$. The alternation in sign is necessary to cause all these vectors to point outward.

Apart from the irrelevant similarity factor v, which may be made unity, the relation between the tetrahedra is thus mutual. The six-fold volume of the second tetrahedron is v^2.

Further, there is this to be observed: The three arrangements *acbd*, *abcd*, *abdc*, distinct by their pairs of opposite sides, are neither cyclically permutable into one another nor cyclic permutations reversed in order. They furnish, in general, three distinct tetrahedra orthogrammatic to the given one and corresponding to the three tetragrams formed with the latter's edges.

The sides of $ABCD$ are ordered into a reversible cycle by the selection of such a tetragram, and since the latter defines its tetrahedron it seems more accurate to speak of orthogrammacy between *tetragrams*, in which case the relation is strictly one-to-one reciprocal.

A second mutual relation between certain pairs of tetrahedra is *orthology*. Steiner[1] introduced the term "orthologiques" for a pair of triangles ABC and $A'B'C'$ so situated that the perpendiculars from A, B, C, on the sides $B'C'$, $C'A'$, $A'B'$ concur (in a point P). It is then proved, commonly by the aid of trilinear coördinates, that this relation is mutual, i.e. that the perpendiculars from A', B', C' to BC, CA, AB are likewise concurrent (in a point O).

It is found that a demonstration of this theorem by vector analysis is directly applicable to tetrahedra in S_3 and further to simplicia [i.e. $(n+1)-$points] in n-dimensional space S_n. We here indicate it for two tetrahedra $ABCD$ and $A'B'C'D'$.

Let the perpendiculars from A', B', C', D' to the sides $a=BCD$, $b=CAD$, $c=ABD$, $d=ABC$ concur in O, which we take as origin. Then, denoting the vectors from O to A, B, C, D, by α, β, γ, $\delta=d_0\alpha+d_1\beta+d_2\gamma$ and those to A', B', C', D', by α', β', γ', δ', we have:

$$\alpha' = a(\delta\times\gamma+\gamma\times\beta+\beta\times\delta) = [(d_1+d_2-1)\beta\times\gamma - d_0\gamma\times\alpha - d_0\alpha\times\beta]a,$$

$$\beta' = b(\delta\times\alpha+\alpha\times\gamma+\gamma\times\delta) = [-d_1\beta\times\gamma + (d_2+d_0-1)\gamma\times\alpha - d_1\alpha\times\beta]b,$$

$$\gamma' = c(\delta\times\beta+\beta\times\alpha+\alpha\times\delta) = [-d_2\beta\times\gamma - d_2\gamma\times\alpha + (d_0+d_1-1)\alpha\times\beta]c,$$

$$\delta' = d(\alpha\times\beta+\beta\times\gamma+\gamma\times\alpha) = [\beta\times\gamma+\gamma\times\alpha+\alpha\times\beta]d.$$

The normals from the vertices A, B, C, D to the sides of A', B', C', D', are given by

$$\alpha + x'(\delta'\times\gamma'+\gamma'\times\beta'+\beta'\times\delta') = \alpha + x[-(cd+bc+bd)\alpha+cd\beta+bd\gamma+bc\delta],$$

$$\beta + y'(\delta'\times\alpha'+\alpha'\times\gamma'+\gamma'\times\delta') = \beta + y[cd\alpha-(da+ac+cd)\beta+da\gamma+ca\delta],$$

$$\gamma + z'(\delta'\times\beta'+\beta'\times\alpha'+\alpha'\times\delta') = \gamma + z[bd\alpha+da\beta-(bd+ab+da)\gamma+ab\delta],$$

$$\delta + t'(\beta'\times\gamma'+\gamma'\times\alpha'+\alpha'\times\beta') = \delta + t[bc\alpha+ca\beta+ab\gamma-(bc+ca+ab)\delta],$$

where x', y', z', t' by their (real) values determine the points on each normal, and $x=x'(1-d_0-d_4-d_2)\,v$, $y=y'(1-d_0-d_4-d_2)\,v$ etc., v again being $[\alpha\beta\gamma]$, the six-fold volume of $ABCD$.

It is now clear that the point

$$(bcd\alpha + cad\beta + abd\gamma + abc\delta)/(bcd + cad + abd + abc)$$

[1] Gergonne's Annales de Mathématiques Pures et Appliquées, vol. 19.

lies on all four normals and corresponds to the particular values

$$x = a/n, \quad y = b/n, \quad z = c/n, \quad t = d/n, \quad \text{where} \quad n = bcd + cad + abd + abc.$$

This proves the theorem that *if the perpendiculars from the vertices of a first tetrahedon to the sides of a second concur in a point, those from the vertices of the second to the sides of the first will likewise concur.*

Using Steiner's expression, two such tetrahedra are *orthologic.*

VI. A Solution of the Quartic Equation

By Raymond Garver, University of California at Los Angeles

It is the purpose of this note to present a solution of the general quartic equation which follows as a direct corollary from two theorems on Tschirnhaus transformations. It does not, however, at all resemble the usual solution by the Tschirnhaus transformation, which requires, first, reduction to the trinomial form $y^4+c_2y^2+c_4=0$ and, secondly, expression of the roots of the original quartic in terms of those of the transformed equation. Nor does it seem to be similar to any of the known solutions.

The two theorems required are included in an article of mine which was recently published in the Messenger of Mathematics.[1] I shall re-state them here, together with an outline of the methods used in their proof. I gave an earlier proof of the first theorem in a paper entitled: *A new normal form for quartic equations*,[2] but it is not well suited to the present purpose.

Theorem I. *Any quartic equation in the form*

$$x^4 + a_2x^2 + a_3x + a_4 = 0 \tag{1}$$

can be transformed, by a quadratic Tschirnhaus transformation

$$y = x^2 + k_1x + k_2, \tag{2}$$

into a quartic whose second term is missing, and which has a double root.

The required transformation has $k_2=a_2/2$, while k_1 may be either square root of any one of the roots of the cubic

$$k^3 + 2a_2k^2 + (a_2^2 - 4a_4)k - a_3^2 = 0. \tag{3}$$

This result is obtained easily. Call the roots of (1) x_1, x_2, x_3, x_4 and those of the transformed equation y_1, y_2, y_3, y_4. Then if we wish, say, y_1 to equal y_2, we may use (2) to obtain a necessary and sufficient condition on k_1. We find $k_1=-(x_1+x_2)=(x_3+x_4)$. Similarly, if $k_1=(x_i+x_j)$, $i, j=1, 2, 3, 4, i\neq j$, the

[1] Vol. 57 (Nov. 1927), pp. 99–101.

[2] Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 310–314.

transformed equation will have a double root. But these six possible values of k_1 are the six square roots of $(x_1+x_2)^2$, $(x_1+x_3)^2$, $(x_1+x_4)^2$, and these, in turn, are the roots of the cubic (3). This cubic can be set up almost immediately with the aid of the elementary symmetric functions of the roots of (1). The reader will also notice that the roots of (3) are four times the roots of the familiar "Euler's cubic" which appears in his solution of the quartic.

The requirement that the transformed equation have a double root imposes no condition on k_2; it is chosen so that the second term in the new equation will vanish.

The second theorem is the following:

THEOREM II. *Any quartic equation in the form* (1) *can be transformed, by a cubic transformation*

$$y = x^3 + k_1x^2 + k_2x + k_3, \tag{4}$$

into a quartic with a triple root. If the given quartic does not have a double root, it is necessary and sufficient that $k_1=x_i$ (*any root of the quartic*), *while* $k_2=x_i^2+a_2$.

The proof is a little longer than that of Theorem I, but is obtained in a similar manner. If we wish to make $y_1=y_2=y_3$, the x_i of the two preceding equations is x_4, and so on.

Consequently, if we allow the solution of the given quartic we may set up transformation (4) directly. However, it seems much more interesting to adopt the following point of view: is it not possible to set up (4) indirectly, and thus arrive at a solution of the quartic by examining the coefficient of x^2? We find that this is possible and simply requires a repeated application of Theorem I.

Assume that (1) does not have a double root, and apply (2) with $k_1=-(x_1+x_2)=(x_3+x_4)$ and $k_2=a_2/2$. Then in the transformed equation we shall have

$$y_1 = y_2 = -x_1x_2 + k_2, \quad y_3 = x_3^2 - (x_1 + x_2)x_3 + k_2, \quad y_4 = x_4^2 - (x_1 + x_2)x_4 + k_2. \tag{5}$$

Now to the equation in y, which we do not need to set up explicitly, apply a second quadratic transformation

$$z = y^2 + c_1y + c_2. \tag{6}$$

If we make $z_1=z_3$, the new equation in z will have a triple root. And this simply requires that we choose $c_1=-(y_1+y_3)=x_4^2-x_3^2$. (To make $z_1=z_4$, choose $c_1=x_3^2-x_4^2$.)

If we substitute (2) into (6), and reduce by means of (1), we have the explicit transformation which reduces (1) to a quartic in z which has $z_1=z_2=z_3$. We require simply the first two terms of the result:

$$z = 2k_1x^3 + (k_1^2 + c_1)x^2 + \cdots \tag{7}$$

Now if the transformation $z=P(x)$ leads to a transformed equation with a triple root, so will the transformation $z=aP(x)$, where a is any constant. That is,

$$z = x^3 + (\tfrac{1}{2}k_1 + \tfrac{1}{2}c_1k_1^{-1})x^2 + \cdots \tag{8}$$

leads to an equation with a triple root; and by Theorem II the coefficient of x^2 is the root x_4 of (1).

By simplifying $\frac{1}{2}c_1k_1^{-1}$, we may write $x_4=(k_1+k_1'+k_1'')/2$, where $k_1=x_k+x_3$, $k_1'=x_4+x_2$, $k_1''=x_4+x_1$, and where these three expressions are square roots of roots of (3). Similarly we see that we have a root x_i provided $k_1=x_i+x_j$; $k_1'=x_i+x_k$; $k_1''=x_i+x_l$; where i, j, k, l are the numbers 1, 2, 3, 4 in some order. To obtain a root, we choose for k_1 either square root of any root of (3); this is of the form x_i+x_j. Choose for k_1' either square root of either of the other roots of (3); this has one of the x's in common with k_1, and is thus of the form x_i+x_k. Now k_1'' is determined uniquely by the condition $(x_i+x_j)(x_i+x_k)(x_i+x_l)=-a_3$. This final process leads to four roots, and is of course essentially the same as that arrived at in most of the other solutions of the quartic.

In our work we assumed that (1) did not have a double root; but since the expression for x_i in the preceding paragraph is simply an identity, it is true for this case as well. In fact, we might now disregard our framework, and present a solution based simply on the preceding paragraph and a knowledge of the form of the roots of (3). This would, it seems, be as short a solution as could possibly be obtained.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED*

College Algebra, by N. R. WILSON and L. A. H. WARREN. Oxford, University Press. viii+451 pages. $2.08.

A text by two members of the faculty of the University of Manitoba.

Precis d'analyse mathematique, par E. LAINÉ, Paris, Librairie Vuibert. Vol. I, 232 pages, 30 fr. Vol. II, 336 pages, 40 fr.

"Preliminaires, fonctions de variables réelles, fonctions analytiques; équations differentielles, géométrie infinitséimale, équations aux dérivées partielles."

Elementary Calculus, by F. S. WOODS and F. H. BAILEY. Revised Edition. Boston, Ginn and Company. x+385 pages. $3.00.

A new edition of a standard and popular text.

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* The quotations given below, unless otherwise specified, are from prefaces and introductions to the books named.

New Analytic Geometry, by P. F. SMITH, A. S. GALE, and J. H. NEELLEY. Revised Edition. Boston, Ginn and Company. $2.00.

A new edition, of a standard and popular text.

The Logic of Modern Physics, by P. W. BRIDGMAN. New York, The Macmillan Company. xiv+228 pages.

"This essay will be largely concerned with the fundamental concepts; it will appear that almost all the concepts can profit from re-examination."

Les espaces abstraits, par M. FRECHET. Paris, Gauthier-Villars. xi+296 pages.

A systematic exposition of the new concepts which constitute the general analysis.

Lecons sur les nombres transfinis, par WACLAW SIERPINSKI. Paris, Gauthier-Villars. vi+249 pages. 40 fr.

"The book offers the reader a clear and easy exposition of the theory of assemblages and to that of the theory of transfinite numbers. To read this book, one has no need of preliminary special studies. . . . . It ought to interest not only the professional mathematician, but also the philosopher."

A History of Mathematical Notations, by FLORIAN CAJORI. Volume I, Elementary Mathematics. Chicago, Open Court Publishing Company. xvi +451 pages. $6.00.

The first volume of a highly significant historical work.

Plane and Spherical Trigonometry, by J. SHIBLI. Boston, Ginn and Company xii+220+94 pages.

A thorough and complete text.

Algebra for Secondary Schools, by S. EMORY and E. E. JEFFS. New York, D. Van Nostrand Company. x+626 pages. $1.85.

This text is distinguished by the very large number of problems serially numbered, and totalling over 11000.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Dynamical Systems. By GEORGE D. BIRKHOFF. American Mathematical Society Colloquium Publications, Volume IX. New York, 1927. viii+295 pages.

According to the author a large part of the material presented in the book is from his colloquium lectures, delivered at the University of Chicago before the American Mathematical Society, September 5–8, 1920. The book is not a treatment of dynamical systems in general but rather it concerns itself with those fields of dynamics that have been of particular interest to the author in the last few years. Particular emphasis is given to the author's own researches

on dynamical systems and a large portion of the references given in the book are found in the author's own papers.

In the first part of Chapter I are given the proofs for a few well known existence, uniqueness, and continuity theorems for normal systems of differential equations. Then the notions of generalized coördinates, generalized external forces, work, and conservative dynamical systems are introduced. Lagrange's equations are then shown to be special cases of certain derived expressions for the generalized forces in a conservative system. The chapter closes with a very interesting "internal and external characterization" of Lagrangian systems and a brief consideration of dissipative systems.

In Chapter II, Hamilton's principle and the principle of least action are derived from Lagrange's equations. A normal form for the principle function for the case of two degrees of freedom is given. The Hamiltonian equations are obtained and compared with the Pfaffian equations. In Chapter III formal aspects are considered. Formal solutions for the equilibrium problem, including the general case in which the time enters explicitly in the differential equations of motion, are obtained. A normal form for both the Hamiltonian and Pfaffian equilibrium problem is also given.

Chapters IV, V, and VI deal almost entirely with periodic motion. In Chapter IV complete stability and other types of stability of periodic motion are considered. Various general methods of establishing the existence of periodic motion are contained in Chapter V, while special methods for the case of one and two degrees of freedom—applying Poincaré's last geometric theorem—are given in Chapter VI.

An attempt at a general theory of dynamical systems giving the types of motion is made in Chapter VII. This is followed, in Chapter VIII, by a very detailed treatment of the case of two degrees of freedom.

The book closes in Chapter IX with a very excellent treatment of the problem of three bodies based on the recent work of Sundman. The reduction in the order of the differential equations is accomplished in the usual manner. The impossibility of triple collision, except under very special conditions, is then established. The author then shows how to obtain indefinite continuation of the motion beyond double collision. Of course, the author here refrains from a rigorous treatment of the question, and the reader is therefore referred to the work of Sundman and Levi-Civita. However, an attempt is made to make the method physically plausible. This is followed by a brief classification of the types of motion in the general case of the problem of three bodies. At the end of the chapter the author indicates a possible generalization to more than three bodies and more general laws of forces.

In view of the many indignities which mechanics has suffered in recent years in the growth of the quantum mechanics, it is a great relief to find in a

book such as Professor Birkhoff has written, that there are still possibilities for the classical mechanics. The large but very interesting amount of research work comprised in this volume merely illustrates that additional hypotheses are not as yet needed if one wishes to make new discoveries in dynamics. Though the book is intended primarily for the research student in theoretical mechanics, any student of mathematics should find it of very great interest. Without doubt the author's desire "that the lectures may serve to stimulate others to investigate the outstanding problems in this most fascinating field" will be fulfilled.

WALTER BARTKY

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3352. Proposed by B. C. Keeler, New York City.

Evaluate

$$\int_0^b \left(\frac{a-\cos x}{\cos x-\cos b}\right)^{1/2} dx, \quad a>1 \text{ and } 0<b<\pi.$$

3353. Proposed by W. H. Roever, Washington University.

Given the three infinite series

$$S_1(x) = x - x^4 + x^9 - x^{16} + \cdots + (-1)^{n+1}x^{n^2} + \cdots,$$

$$S_2(x) = \frac{x\sin y}{y} - \frac{x^4\sin 2y}{2y} + \frac{x^9\sin 3y}{3y} + \cdots + (-1)^{n+1}\frac{x^{n^2}\sin ny}{ny} + \cdots,$$

$$S_3(x) = x\cos y - x^4\cos 2y + x^9\cos 3y + \cdots + (-1)^{n+1}x^{n^2}\cos ny + \cdots,$$

show that $S_1(x)<1/2$, when $0<x<1$; $S_2(x)<1/2$; $S_3(x)\leqq 1/2$, when $0<x<1$ and $0<y<\pi$.

These problems arose in an investigation on the structure of the atom.

3354. Proposed by J. V. Uspensky.

If p be a prime of the form $4l+3$, then

$$f(l) = 1 - \left(\frac{l}{1!}\right)^2\frac{1}{8} + \left(\frac{l(l-1)}{2!}\right)^2\frac{1}{8^2} - \cdots \equiv 0 \pmod{p}.$$

But if p be a prime of the form $4l+1$, then $f(l)\equiv 2^l C_{2l}{}^l \pmod{p}$.

3355. Proposed by Elmer Schuyler, Brooklyn, N. Y.

In the *Plane and Solid Geometry* by Wentworth-Smith (revised edition) occurs the following propo-

sition: Of all polygons with sides all given but one (and in a definite order), the maximum can be inscribed in a semicircle which has the undetermined side for its diameter.

Prove that there is one and only one maximum. Given the lengths of the sides and their order, compute the radius of the semicircle.

3356. Proposed by L. S. Johnston, Pennsylvania State College.

Consider a triangle ABC, with angles at A and B acute and the angle at A greater than the angle at B. Let A', B', and C' be the feet of the respective perpendiculars from A, B, and C to the opposite sides. Let the perpendiculars from A to the lines $A'B'$ and $A'C'$ meet those lines at R_1 and R_2, respectively; let the perpendiculars from B to the lines $B'C'$ and $B'A'$ meet those lines at S_1 and S_2, respectively; let the perpendiculars from C to the lines $C'A'$ and $C'B'$ meet those lines at T_1 and T_2, respectively. Show that the conic with A' and B' as foci and S_2 and R_1 as vertices is tangent to AB at C'; that the conic with B' and C' as foci and T_2 and S_1 as vertices is tangent to BC at A'; that the conic with C' and A' as foci and R_2 and T_1 as vertices is tangent to CA at B'.

3357. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Through a given point to draw a line so that the segment intercepted on it by two given intersecting lines shall be divided in a given ratio by the foot of the perpendicular dropped upon it from the point of intersection of the two given lines.

3358. Proposed by J. V. Uspensky.

Let p be a prime such that $p\equiv 5 \pmod 6$ and $p=4l+1$ or $p=4l+3$. Show that in this case

$$f(l) = 1 + \frac{1}{4}\left(\frac{l}{1}\right)^2 + \frac{1}{4^2}\left(\frac{l(l-1)}{1\cdot 2}\right)^2 + \cdots \equiv 0 \pmod 6.$$

Show also that if $p=4l+1=12n+1$,

$$f(l) \equiv (-1)^n \cdot C_{6n}^{3n} \cdot C_{6n}^{3n}/C_{6n}^{n} \pmod p;$$

and that if $p=4l+3=12n+7$,

$$f(l) \equiv (-1)^{n+1} \cdot 2 \cdot C_{6n+3}^{3n+1} \cdot C_{6n+3}^{3n+1}/C_{6n+3}^{n} \pmod p.$$

3359. Proposed by Clifford N. Mills, Normal, Illinois.

A. B, C, D are four points in a plane. Let the centroids of the triangles BCD, CDA, DBA, ABC be respectively a, b, c, d. Then Aa, Bb, Cc, Dd meet in a point, and are divided in the same ratio at this point. Also the quadrilateral $abcd$ is similar to $ABCD$.

3360. Proposed by Bernardo Ig. Baidaff, Buenos Aires.

Solve the system $x_i=a^i+bx_{i-1}+b^2x_{i-2}$ $(i=1, 2, \cdots, n)$, with $x_{-1}=x_0=0$. Generalize.

SOLUTIONS

446 [1914, 191]. Proposed by S. G. Barton, University of Pennsylvania.

Prove any one or more of the sixteen theorems, stated without proof, in the article in this issue of the MONTHLY (June, 1914, pages 182–184) on "Properties of the Normals to a Conic."

SOLUTION BY OTTO DUNKEL, Washington University

Theorem (1) gives the equations of the coincident and single normals to the parabola $y^2=4px$ from a point (a, b), the coördinates of the intersections of these normals with the curve, and some facts obvious by inspection of these results. These results will be derived.

If x_1, y_1 is a point on the parabola, we easily find the slope of the normal to be $m=-y_1/2p$, $x_1=pm^2$, $y_1=-2pm$. The equation of the normal is then $pm^3+(2p-x)m+y=0$. Suppose that two normals through (a, b) coincide and have the slope m_1, while the third (single) normal through the same point has the slope m_2. Then from the above equation we find $m_2=-2m_1$, $m_1^2+2m_1m_2=(2p-a)/p$, $m_1^2m_2=-b/p$. From the first and third of these we get $m_1=(b/2p)^{1/3}$; from the first and second, $3m_1^2=(a-2p)/p$; and

finally $(b/2p)^2=[(a-2p)/3p]^3$. Hence $m_1=[(a-2p)/3p]^{1/2}$, $m_2=-2m_1$, where b is taken as positive. The point on the curve corresponding to (a, b) for the slope m_1 is $(pm_1^2, -2pm_1)$, and similarly for m_2. Hence the equation of the double normal and the coördinates of the corresponding point on the curve are

$$y-b=\left(\frac{a-2p}{3p}\right)^{1/2}(x-a), \quad \left[\frac{a-2p}{3}, \quad -2\left(\frac{(a-2p)p}{3}\right)^{1/2}\right],$$

while for the single normal they are

$$y-b=-2\left(\frac{a-2p}{3p}\right)^{1/2}(x-a), \quad \left[\frac{4}{3}(a-2p), \quad 4\left(\frac{(a-2p)p}{3}\right)^{1/2}\right].$$

478 [1915, 338]. Proposed by J. A. Campbell, St. Johnsbury, Vt.

Construct a triangle having given the difference of the segments of the base formed by the foot of the perpendicular from the vertical angle, the difference of the base angles, and the sum of the three sides.

523 [1917, 426]. Proposed by H. Campbell, St. Johnsbury, Vt.

Given the difference of the segments of the base made by the perpendicular let fall from the vertical angle, the difference of the base angles, and the sum of the sides of a triangle, to construct the triangle.

3324 [1928, 261]. Proposed by H. Campbell, St. Johnsbury, Vt.

Given the difference of the segments of the base made by the perpendicular to the base from the vertical angle, the difference of the base angles, and the difference of the sides including the vertical angle, to construct the triangle.

Solution by Nathan Altshiller-Court, University of Oklahoma

Let ABC be the required triangle and O its circumcenter. The parallel through A to BC meets the circumcircle (O) in A'. It is readily seen that the segment AA' is equal to the difference, d, of the segments into which the foot of the altitude from A divides BC; hence $AA'=d$ is known. Since $A'C=AB$, we have angle $ABA'=$angle $ABC-$angle $A'BC=$angle $ABC-$angle ACB. But the difference of the last two angles is given; hence the angle ABA' is known. Thus in the circle (O) we know the chord AA', and the angle ABA' inscribed, which this chord subtends; hence the circle (O) is known. Suppose then that this circle is drawn and that the chord AA' is placed in it.

Let $EF=2R$ be the diameter of (O) perpendicular to BC, the points A and E being assumed to lie on opposite sides of BC. This diameter is also perpendicular to AA'; hence the points E, F are known. The line AE is therefore known both in magnitude and position.

The point E is the center of a circle (E) passing through B, C and cutting AE in the incenter I and in the excenter I' relative to the side BC. The perpendicular $I'X$ from I' to AB determines on AB a segment AX equal to half the sum of the sides of the triangle ABC. The inscribed angles BAE, BFE being equal as subtending the same chord, the right triangles $AI'X$, BEF are similar; hence

$$AX/BF=AI'/EF. \tag{1}$$

If we denote by x the radius of the circle (E), by t the known segment AE, and by $2p$ the given perimeter of the triangle ABC, we have $AX=p$, $EF=2R$, $AI'=t+x$, $BF^2=4R^2-x^2$. The equation (1), when rationalized, becomes

$$(t+x)^2(4R^2-x^2)=4R^2p^2,$$

where p, R, and t are known segments. The radius x of (E) thus being determined the circle (E) is known. The common chord of the two circles (O), (E) is the base BC of the required triangle ABC.

Since the radius, x, of (E) depends upon an irreducible biquadratic equation, the proposed problem cannot be solved by ruler and compasses.

Note by Otto Dunkel, Washington University

Setting $\angle ABA'=\delta$, we have $t=2R\cos(\delta/2)$, $x=2R\sin(A/2)$, and the above equation becomes

$$(p/2R)=u=[\cos(\delta/2)+\sin(A/2)]\cos(A/2), \quad 0\leqq A\leqq 180^\circ-\delta.$$

We find $2\,du/dA = 1-\cos(\delta/2)\sin(A/2)-2\sin^2(A/2)$, and the right side of this equation has the values 1 and $1-3\cos^2(\delta/2)$ at the ends of the range for A. Hence, if $\cos(\delta/2) \leqq 3^{-1/2}$, u increases from $\cos(\delta/2)$ to $\sin\delta$. The value of u must be taken in this range and there is then one and only one solution. If $\cos(\delta/2) > 3^{-1/2}$, then u takes on a maximum value u_m given by A_m, where

$$4\sin(A_m/2) = [8+\cos^2(\delta/2)]^{1/2} - \cos(\delta/2). \tag{1}$$

It is easily seen that A_m is greater than 60°. Hence the value of u in this case must be selected from the range passing from the smaller of $\cos(\delta/2)$ and $\sin\delta$ to the maximum, u_m. Thus there may be one or two solutions according to the position of u in this range, and the two cases may be easily separated. Setting $z=\cos(A/2)$, $c=\cos(\delta/2)$, $s=\sin(\delta/2)$, we find

$$z^4 - s^2z^2 - 2cuz + u^2 = 0. \tag{2}$$

This equation may be replaced by two quadratics by means of a root of the equation

$$t^3 + s^2t^2 - 4u^2t - 4u^2 = 0. \tag{3}$$

It may be shown that this equation has in general no root which is rational in u^2 and s^2, and therefore there is no ruler and compass construction for the triangle. The proof will be given for a special case where $\cos(\delta/2)=\sin\delta$, i.e. $\delta=60°$. Here $u_m=1.18391$, and u may have any value such that $3^{1/2}/2 \leqq u \leqq 1.18391$. We shall set $u=1$ and there will be two solutions. Equation (3) is then $4t^3+t^2-16t-16=0$, and its only real root is between 2 and 3. This root cannot be rational, for the only rational roots that this equation can have must be powers of 2 or the reciprocals of such powers. Hence the two triangles cannot be constructed by ruler and compass. Since (3) has only one real root, the equation (2) has only two: .600317 and .989491. The two triangles are then

A	B	C
106°12.9′,	66°53.55′,	6°53.55′,
16°37.6′,	111°41.2′,	51°41.2′,

where $p=2R$, $AA'=d=3^{1/2}R$.

If from E we drop the perpendicular EY to AB, then $2AY=b+c$, where $b=AC$, $c=AB$; also $2BY=b-c$. Hence, if in place of giving the value of $2p$, we have given $c+b=2k$, the construction for ABC is as follows: Draw a circle with AE as diameter, and with A as center draw a circle of radius k cutting the first circle in Y and Y'. Then AY and AY' cut the circle (O) in B and C. No construction is possible if $d<2k\sin(\delta/2)$.

If $b-c=2l$ is given, draw AO cutting the circle (O) again in G. Construct the circle with EG as diameter, and with E as center describe a circle with radius l cutting the first circle in Z and Z'. Then GZ and GZ' cut the circle (O) in B and C. No construction is possible if $d<2l\cos(\delta/2)$.

Note by the Editors

A solution of problem 523 in which "sum of the sides" is interpreted as meaning the sum of the two sides which include the vertical angle is as follows:

Solution by Laurence Hampton, University of Oklahoma

Assume that the triangle ABC has been constructed. Draw the altitude from B, meeting AC at D. Let p and q represent the segments AD and DC. Let p be larger than q. Then on DA lay off the distance DE equal to q. Then AE is equal to $p-q$ which is known. Extend AB to F so that BF is equal to BC. Then AF is equal to $c+a$ which is given. The triangle BEC is isosceles and angle BEC is equal to C. But it is an exterior angle of the triangle ABE and, therefore, angle ABE is $C-A$. But this is an exterior angle of the isosceles triangle BEF. Therefore, BFE is equal to $\frac{1}{2}(C-A)$. The triangle AFE can be constructed since we have two sides and one of its angles given.

Construct the triangle AFE. Find the intersection of AF with the perpendicular bisector of FE, thus locating B. With B as center and BE as radius draw arc cutting AE produced at C. Then ABC is the required triangle.

The triangle AFE is the ambiguous case of triangles. If $p-q<(a+c)\sin\frac{1}{2}(C-A)$ it cannot be constructed. If $p-q$ is greater than this there will be two possibilities for AFE. Call the second triangle AFE'. Call the resulting solution $AB'C'$. Locate N, the mid-point of EE'. The perpendicular bisector of FN will be equidistant from the perpendicular bisectors of FE and FE' which determined B and B'. It will, therefore, bisect BB' in a point M. But M is the midpoint of AF and therefore AB' equals BF which was equal to BC. By considering right triangles AFN and ANE we can show that angle AEN is $\frac{1}{2}(A+C)$. Then CAC' is $A+C$ and $B'AC'$ is, therefore, equal to C. Then C' is equal to A. The two solutions are, therefore, congruent.

2695 [1918, 170]. Proposed by Frank Irwin, University of California.

A positive number, which for convenience we will write as a fraction, a/b, is developed into a continued fraction,

$$a_1 \frac{1}{-a_2} \frac{1}{-a_3} \cdots,$$

by the following process:

$$a = a_1 b - r_1, \qquad b = a_2 r_1 - r_2, \qquad r_1 = a_3 r_2 - r_3, \text{ etc.}$$

Here $a_1, a_2, \cdots$ are positive integers, and, if we suppose a, b to have been taken positive, $b>r_1>r_2>\cdots>0$. Show that if a/b be but slightly larger than a positive integer or zero, the numerators (as likewise the denominators) of the successive convergents will for a time be in arithmetical progression, and determine how long this phenomenon will continue.

Solution by Charles H. Smiley.

By the conditions of the problem, $a/b=a_1-1+\epsilon_1$, where a_1 is a positive integer equal to or greater than one, and ϵ_1 is a positive quantity less than unity. We may write

$$\begin{aligned} a/b &= a_1 - (1-\epsilon_1) = a_1 - 1/(1+\epsilon_2), \quad \epsilon_2 > \epsilon_1, \\ &= a_1 - \frac{1}{2-(1-\epsilon_2)} = a_1 - \frac{1}{2-\dfrac{1}{1+\epsilon_2}}, \quad \epsilon_3 > \epsilon_2, \\ &= a_1 - \frac{1}{2-\dfrac{1}{2-(1-\epsilon_3)}}, \end{aligned}$$

and so on until we reach $\epsilon_i \geqq 1$.

In this case $0<1-\epsilon_{i-1}\leqq 1/2$, $1>\epsilon_{i-1}\geqq 1/2$, and we find in turn

$$1 > \epsilon_{i-1} \geqq 1/2 > \epsilon_{i-2} \geqq 1/3 > \epsilon_{i-3} \geqq 1/4 \cdots 1/(i-1) > \epsilon_1 \geqq 1/i.$$

Conversely, if $1/(i-1)>\epsilon_1\geqq 1/i$, we find that $\epsilon_i \geqq 1$ and that

$$a_2 = a_3 = a_4 = \cdots = a_{i-1} = 2, \quad a_i > 2,$$

and for the successive convergents, we have

$$\begin{aligned} k &= 0 \quad 1 \quad\quad 2 \quad\quad\quad \cdots (i-2), \\ P &= a_1 \quad (2a_1-1) \quad (3a_1-2)\cdots(i-1)a_1-(i-2), \\ Q &= 1 \quad 2 \quad\quad 3 \quad\quad\quad \cdots (i-1). \end{aligned}$$

We find that under the conditions named above, the numerators of the first $(i-1)$ convergents are in arithmetical progression, common difference, (a_1-1). Also the denominators are the first $(i-1)$ integers and, therefore, are in arithmetical progression. In case $\epsilon_1=1/i$, we find that, in addition, $a_i=2$ and hence the numerators (as well as the denominators) of the first (i) convergents are in arithmetical progression.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor G. D. BIRKHOFF of Harvard University delivered a lecture on "The Mathematical Basis of Art" at the Ohio State University on December 3, 1928.

Professor E. R. HEDRICK, of the University of California at Los Angeles, filled the following schedule of lectures during November and December, 1928: University of Missouri, November 26; Ohio State University, November 27–28; University of Cincinnati, November 30–December 3; University of Kentucky, December 4; Oberlin College, December 5–6; University of Michigan, December 7; University of Wisconsin, December 10; Northwestern University, December 11; University of Indiana, December 12; University of Pittsburgh, December 13–17; University of West Virginia, December 18; University of Pennsylvania, December 19.

At the thirteenth annual meeting of the Optical Society of America, held in Washington, October 31 to November 3, 1928, under the joint auspices of that society and the Bureau of Standards, was celebrated the semicentennial of the publication of Professor A. A. Michelson's first communication on the velocity of light. On this occasion, Professor Michelson and his collaborators, Messrs. Pease and Pearson, delivered an address on *Results of repetition of the Michelson-Morley experiment.*

Professor RAYMOND PEARL, of Johns Hopkins University, has been elected president of the International Union for the Study of Demographic Problems.

The following have been appointed to National Research Fellowships in mathematics for 1928–29; A. A. Albert, W. L. Ayres, F. R. Bamforth, Theodore Bennett, Alonzo Church, T. F. Cope, C. M. Cramlet, Jesse Douglas, D. A. Flanders, J. J. Gergen, M. S. Knebelman, Lincoln La Paz, Morris Marden, T. W. Moore, Hillel Poritsky, N. E. Rutt, I. M. Sheffer, Marie J. Weiss. Messrs. Ayres, Church, and Douglass hold fellowships supported by the International Education Board.

Dr. H. A. BENDER of the University of Illinois, has been appointed assistant professor of mathematics at the Municipal University of Akron, Ohio.

Mr. W. M. BOND has been appointed professor of mathematics at Waynesburg College, Waynesburg, Pa.

Mr. P. T. COPP has been appointed head of the department of mathematics at Valparaiso University.

Professor TOMLINSON FORT, of Lehigh University, paid a visit to the mathematics clubs of Rutgers University on November 9 and delivered a lecture on "The place of Sturm's theorem in algebra."

Dr. MARY G. HASEMAN has been appointed professor of mathematics at Hartwick College, Oneonta, N. Y.

Associate Professor B. V. HILL, of Phillips University, Enid, Oklahoma, has been promoted to a professorship of mathematics.

Associate Professor L. S. JOHNSTON, of Pennsylvania State College, has been appointed head of the department of engineering mathematics at the University of Detroit.

Dr. O. J. PETERSON has been appointed head of the department of mathematics at the State Teachers' College at Emporia, Kansas.

Assistant Professor J. B. SCARBOROUGH, of the United States Naval Academy, has been promoted to an associate professorship of mathematics.

Dr. S. A. SCHELKUNOFF, of the State College of Washington, has been promoted to an assistant professorship of mathematics.

Professor A. A. TITSWORTH has retired after forty-two years of service at Rutgers University.

Associate Professor MARY E. WELLS, of Vassar College, has been promoted to a professorship of mathematics.

Associate Professor C. H. Yeaton, of Oberlin College, has been promoted to a professorship of mathematics.

The following appointments to instructorships in mathematics are announced:

George Washington University, Mr. Edgar W. Woolard of the United States Weather Bureau;

University of Michigan, Mr. P. M. SWINGLE;

Northwestern University, Mr. R. H. BARDELL;

Pennsylvania State College, Messrs. L. T. DUNLAP, W. O. GORDON, J. A. HAMILTON, P. V. KUNKEL, and R. E. PETERSON;

St. John's College, Mr. G. C. VEDOVA.

Professor T. A. MARTIN, Berea College, Berea, Kentucky, died on September 11, 1928 after a lingering illness. He had been head of the department of mathematics for seven years and had been a member of the Mathematical Association since December, 1925.

Miss MARION REILLY, of Bryn Mawr College, died in January, 1928. Miss Reilly has been a member of the American Mathematical Society since 1913.

Professor C. D. RICE, of the University of Texas, died on May 28, 1928, at the age of sixty-five. Professor Rice has been a member of the American Mathematical Society since 1920.

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ADDENDA AND CORRIGENDA

P. 37, first line, replace MIAMI by OXFORD.
P. 97, 21st line, replace C by G.
P. 97, 2nd line, from bottom, replace F. by W. F.
P. 114, 8th line, replace D. H. by D. N.
P. 127, 5th line from bottom, replace "it" by "its".
P. 134, 24th line, replace $f(t)dt(t-z)$ by $f(t)dt/(t-z)$.
P. 217, replace Thomson by Thompson.
P. 279, 2nd line, replace "bi-orthoganal" by "bi-orthogonal."
P. 335, 8th line, replace "The words were written by Miss Lutz" by "The pageant was written by Miss Banes, assisted by five students of her class in the history of mathematics."
P. 361, 19th line, replace 7.973; by 7.972; 4th line from bottom, replace -5.011 by -5.010.
P. 389, 3rd line, for "sets" read "nets."
P. 399, 5th line, replace J. E. by J. A.
P. 437, in the footnote, replace vol. 33 by vol. 34.
P. 456, 13th line, replace D. by P.
P. 492, 21st line, replace "initmate" by "intimate."

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Thirteenth Annual Meeting, New York City, December 28-29, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.
INDIANA, Butler University, May 11-12.
IOWA, Grinnell College, May 4-5.
KANSAS, Topeka, Kan., February 4.
KENTUCKY, Louisville, Ky., April 21.
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.
MICHIGAN, Ann Arbor, Mich., March 31.
MINNESOTA, St. Joseph, Minn., May 19.
MISSOURI, Kansas City, Mo., November.
NEBRASKA, Midland College, April 28.
OHIO, Columbus, Ohio, April 5.
PHILADELPHIA, Philadelphia, Pa., December 1.
ROCKY MOUNTAIN, Golden, Colo., April 20-21.
SOUTHEASTERN, Dunham, N. C., March.
SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.
TEXAS, Texas A. and M. College, Jan. 28.

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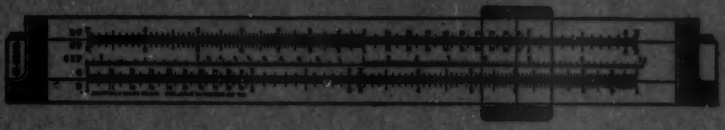

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